

ORBIT CONFIGURATION SPACES OF SMALL COVERS AND QUASI-TORIC MANIFOLDS

JUNDA CHEN, ZHI LÜ AND JIE WU

ABSTRACT. In this article, we investigate the orbit configuration spaces of some equivariant closed manifolds over simple convex polytopes in toric topology, such as small covers, quasi-toric manifolds and (real) moment-angle manifolds; especially for the cases of small covers and quasi-toric manifolds. These kinds of orbit configuration spaces are all non-free and noncompact, but still built via simple convex polytopes. We obtain an explicit formula of Euler characteristic for orbit configuration spaces of small covers and quasi-toric manifolds in terms of the h -vector of a simple convex polytope. As a by-product of our method, we also obtain a formula of Euler characteristic for the classical configuration space, which generalizes the Félix-Thomas formula. In addition, we also study the homotopy type of such orbit configuration spaces. In particular, we determine an equivariant strong deformation retract of the orbit configuration space of 2 distinct orbit-points in a small cover or a quasi-toric manifold, which turns out that we are able to further study the Betti numbers and (equivariant) cohomology of such an orbit configuration space.

1. INTRODUCTION

Let G be a topological group and let M be a G -space. The (ordered) orbit configuration space $F_G(M, n)$ is defined by

$$F_G(M, n) = \{(x_1, \dots, x_n) \in M^{\times n} \mid G(x_i) \cap G(x_j) = \emptyset \text{ for } i \neq j\}$$

with subspace topology, where $n \geq 2$ and $G(x)$ denotes the orbit at x . In the case where G acts trivially on M , the space $F_G(M, n)$ is the classical configuration space denoted by $F(M, n)$.

The notion of configuration space had been introduced in physics in 1940s [N, vT] concerning the topology of configurations with various study on this important object since then. In mathematics, configuration spaces were first introduced by Fadell and Neuwirth [FN] in 1962 with various applications [Ar, BCWW, Bir, Bo, Co, T, V]. Since 1990s, the notion of configurations was introduced in robotics community to study safe-control problems of robots. Mathematically the topological robotics was recently established by Ghrist and Farber [Fa, Gh], where the topology of configuration spaces on graphs plays an important role in this new created area. The orbit configuration spaces with labels provide combinatorial models for equivariant

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loop spaces [X]. Moreover orbit configuration space is an analogy to fiber-type arrangements [Coh]. The fundamental groups of orbit configuration spaces enrich the theory of braids [CKX, CX].

If the group G acts properly discontinuously on a manifold M , there are various fibrations available related to $F_G(M, n)$ [X]. Thus the standard methods of spectral sequences in algebraic topology can be used for studying the cohomology of $F_G(M, n)$. In particular, the cohomology of $F_{\mathbb{Z}_2}(S^k, n)$ has been determined in [FZ, X2], where \mathbb{Z}_2 acts (freely) on S^k by antipodal map. However the determination of the homotopy type or cohomology of $F_G(M, n)$ become much harder provided that the group G does not act freely on a manifold M because of lacking effective tools from algebraic topology. For instance, the classical Fadell-Neuwirth fibration [FN] fail in non-free cases in general.

In 1991, Davis and Januszkiewicz [DJ] introduced four classes of particularly nicely behaving manifolds over simple convex polytopes—small covers, quasi-toric manifolds and (real) moment-angle manifolds, which have become important objects in toric topology. Note that (real) moment-angle manifolds were named by Buchstaber and Panov [BP] later when they studied the topology of (real) moment-angle manifolds as submanifolds in polydisks. A *quasi-toric manifold* (resp. *small cover*), as the topological version of a compact non-singular toric variety (resp. real toric variety), is a smooth closed manifold M of dimension $2m$ (resp. dimension m) with a locally standard action of torus T^m (resp. real torus \mathbb{Z}_2^m) such that its orbit space is a simple convex m -polytope P . A *(real) moment-angle manifold* can directly be constructed from a simple convex polytope P such that it admits an action of real torus or torus with P as its orbit space. There are strong links between topology and geometry of these equivariant manifolds and combinatorics of polytopes. In this article, we put these equivariant manifolds into the framework of orbit configuration spaces, especially for the cases of small covers and quasi-toric manifolds. In other words, we pay our much more attention on non-free orbit configuration space $F_{G_d^m}(M, n)$ for a dm -dimensional G_d^m -manifold $\pi_d : M \rightarrow P$ over a simple convex m -polytope P , $d = 1, 2$, where M is a small cover and $G_1^m = \mathbb{Z}_2^m$ when $d = 1$, and a quasi-toric manifold and $G_2^m = T^m$ when $d = 2$. We still expect that there is an essential connection between topology and geometry of $F_{G_d^m}(M, n)$ and combinatorics of P .

Our first result is an explicit formula for the Euler characteristic of $F_{G_d^m}(M, n)$ in terms of the h -vector (h_0, h_1, \dots, h_m) of P , and in particular, $\chi(F_{T^m}(M, n)) = \chi(F(M, n))$ if $d = 2$. Let $\mathbf{h}_P(t) = h_0 + h_1 t + \dots + h_m t^m$ be a polynomial in $\mathbb{Z}[t]$. Then our result is stated as follows.

Theorem 1.1. *Let $\pi_d : M \rightarrow P$ be a dm -dimensional G_d^m -manifold over a simple convex m -polytope P where $d = 1, 2$. Then the Euler characteristic of $F_{G_d^m}(M, n)$ is*

$$\chi(F_{G_d^m}(M, n)) = \begin{cases} (-1)^{mn} \sum_{I=(n_1, \dots, n_s)} \mathcal{C}_I \prod_{k=1}^s \mathbf{h}_P(1 - 2^{n_k}) & \text{if } d = 1 \\ \chi(F(M, n)) = \sum_{I=(n_1, \dots, n_s)} \mathcal{C}_I \mathbf{h}_P(1)^s & \text{if } d = 2 \end{cases}$$

where $I = (n_1, \dots, n_s)$ runs over all partitions of n , $\mathcal{C}_I = \frac{n!(-1)^{n-s}}{r_1!r_2!\dots r_s!n_1n_2\dots n_s}$ and r_k is the time number that n_k appears in I .

Our method for proving this theorem is to investigate the combinatorial structure on $F_{G_d^m}(M, n)$. As a consequence of this method, we can also give a formula for the Euler characteristic of a non-equivariant configuration space $F(M, n)$ in terms of a polynomial of $\chi(M)$.

Theorem 1.2. *Let M be a compact triangulated homology m -manifold. Then*

$$\chi(F(M, n)) = (-1)^{mn} \prod_{k=0}^{n-1} (\chi(M) - k) = (-1)^{mn} n! \binom{\chi(M)}{n}.$$

Remark 1. The above formula can be rewritten as

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^{mn} \chi(F(M, n))}{n!} t^n = 1 + \sum_{n=1}^{\infty} \binom{\chi(M)}{n} t^n = (1+t)^{\chi(M)}.$$

Let m be even. Then we obtain the Félix-Thomas formula [FT, Theorem B]. Hence Theorem 1.2 generalizes the Félix-Thomas formula.

Next we shall be concerned with the homotopy type of $F_{G_d^m}(M, n)$ for $m \geq 1$. We first consider the case $m = 1$. In this case, we obtain

Theorem 1.3. *Let $\pi_d : M \rightarrow P$ be a d -dimensional G_d^1 -manifold over P . Then, when $d = 1$, $F_{\mathbb{Z}_2}(M, n)$ has the same homotopy type as $n!2^{n-2}$ points, and when $d = 2$, $F_{S^1}(M, n)$ has the same homotopy type as a disjoint union of $n!$ copies of T^{n-2} .*

Remark 2. The classical configuration spaces on the circle is related to hyperbolic Dehn fillings [YNK]. The orbit version might give some additional information.

For general M and n , the spaces $F_{G_d^m}(M, n)$ can be expressed as an intersection of the subspaces of $M^{\times n}$ which are homeomorphic to $M^{\times(n-2)} \times F_{G_d^m}(M, 2)$ under coordinate permutations (see Proposition 2.1). Thus the study on $F_{G_d^m}(M, 2)$ is the first step for the general cases. In this article we focus on this case to give an experimental investigation about the homotopy type of $F_{G_d^m}(M, 2)$, where the combinatorial methods successfully overcome the technical difficulties in this case. The spaces $F_{G_d^m}(M, n)$ for general n will be explored in our subsequent work. By the reconstruction of small covers and quasi-toric manifolds, we are able to determine an equivariant strong deformation retract of $F_{G_d^m}(M, 2)$ in terms of the combinatorial data from P via π_d . The result is stated as follows.

Theorem 1.4. *Let $\pi_d : M \rightarrow P$ be a dm -dimensional G_d^m -manifold over a simple convex polytope P . Then there is an equivariant strong deformation retraction of $F_{G_d^m}(M, 2)$ onto*

$$X_d(M, 2) = \bigcup_{\substack{F_1, F_2 \in \mathcal{F}(P) \\ F_1 \cap F_2 = \emptyset}} (\pi_d^{-1})^{\times 2}(F_1 \times F_2)$$

where $\mathcal{F}(P)$ is the set of all faces of P .

The equivariant strong deformation retract in Theorem 1.4 plays an important role on further studying the algebraic topology of $F_{G_d^m}(M, 2)$. As a result, we obtain that

Theorem 1.5. *Given a simple convex polytope P , assume that $\pi_d : M \rightarrow P$ is a dm -dimensional G_d^m -manifold over P . Then*

$$b_{2i}(F_{T^m}(M, 2)) = b_i^{\mathbb{Z}_2}(F_{\mathbb{Z}_2^m}(M, 2)),$$

which only depends upon the combinatorial structure of P , where $b_i^{\mathbb{Z}_2}(F_{\mathbb{Z}_2^m}(M, 2))$ is the i -th mod 2 Betti number of $F_{\mathbb{Z}_2^m}(M, 2)$, and $b_i(F_{T^m}(M, 2))$ is the i -th Betti number of $F_{T^m}(M, 2)$. In particular, the homology of $F_{T^m}(M, 2)$ vanishes in odd dimensions and is free abelian in even dimensions.

As a further application of Theorem 1.4, the equivariant cohomology of the orbit configuration spaces $F_{T^m}(M, 2)$ can be also determined as follows:

Theorem 1.6. *Let $\pi_2 : M \rightarrow P$ be a $2m$ -dimensional quasi-toric manifold over a simple convex polytope P . Then the Leray–Serre spectral sequence of the fibration*

$$ET^{2m} \times_{T^{2m}} F_{T^m}(M, 2) \rightarrow BT^{2m}$$

with fiber $F_{T^m}(M, 2)$ collapses at the E_2 term, that is, $E_\infty = E_2$.

Furthermore, as a consequence of Theorems 1.4–1.5 and the Mayer–Vietoris spectral sequence, we can also determine:

- (1) the integral homology of $F_{G_d^2}(M, 2)$ and
- (2) the (mod 2) homology of $F_{G_d^m}(M, 2)$ for M to be a G_d^m -manifold over an m -simplex Δ^m .

The article is organized as follows. In section 2, we give a brief review on the notions of small covers, quasi-toric manifolds and (real) moment-angle manifolds and investigate basic constructions and properties of their orbit configuration spaces. We calculate the Euler characteristic of the orbit configuration spaces for small covers and quasi-toric manifolds in section 3, where Theorem 1.1 is Theorem 3.1 and the proof of Theorem 1.2 is given in subsection 3.4. In section 4, we study the homotopy type of $F_{G_d^1}(M, n)$ and $F_{G_d^m}(M, 2)$ with giving the proofs of Theorems 1.3 and 1.4. As an application of Theorem 1.4, we study the Betti numbers and (equivariant) cohomology of $F_{G_d^m}(M, 2)$ and prove Theorems 1.5 and 1.6 in section 5. In section 6, we compute the integral homology of $F_{G_d^2}(M, 2)$ and the (mod 2) homology of $F_{G_d^m}(M, 2)$ for the G_d^m -manifold M over an m -simplex. The Mayer–Vietoris spectral sequence will be one of major tools for our computations and so we give a review on the Mayer–Vietoris spectral sequence in section 7 as an appendix.

2. G_d^m -MANIFOLDS AND (REAL) MOMENT-ANGLE MANIFOLDS OVER SIMPLE CONVEX POLYTOPES AND THEIR ORBIT CONFIGURATION SPACES

2.1. G_d^m -manifolds and (real) moment-angle manifolds over simple convex polytopes. Following [DJ], let P be a simple convex m -polytope, and let G_d^m be the 2-torus \mathbb{Z}_2^m of rank m if $d = 1$, and the torus T^m of rank m if $d = 2$. A dm -dimensional G_d^m -manifold over P , $\pi_d : M \rightarrow P$, is a smooth closed dm -dimensional manifold M with a locally standard G_d^m -action such that the orbit space is P . A G_d^m -manifold $\pi_d : M \rightarrow P$ is called a *small cover* if $d = 1$ and a *quasi-toric manifold* if $d = 2$. We know from [DJ] that each G_d^m -manifold $\pi : M \rightarrow P$ determines a characteristic function λ_d on P , defined by mapping all facets (i.e., $(m - 1)$ -dimensional faces) of P to nonzero elements of R_d^m such that m facets

meeting at each vertex are mapped to a basis of R_d^m where $R_d = \begin{cases} \mathbb{Z}_2 & \text{if } d = 1 \\ \mathbb{Z} & \text{if } d = 2. \end{cases}$

Conversely, the pair (P, λ_d) can be reconstructed to the M as follows: first λ_d gives

the following equivalence relation \sim_{λ_d} on $P \times G_d^m$

$$(2.1) \quad (x, g) \sim_{\lambda_d} (y, h) \iff \begin{cases} x = y, g = h & \text{if } x \in \text{int}(P) \\ x = y, g^{-1}h \in G_F & \text{if } x \in \text{int}F \subset \partial P \end{cases}$$

then the quotient space $P \times G_d^m / \sim_{\lambda_d}$ is equivariantly homeomorphic to the M , where G_F is explained as follows: for each point $x \in \partial P$, there exists a unique face F of P such that x is in its relative interior. If $\dim F = k$, then there are $m - k$ facets, say $F_{i_1}, \dots, F_{i_{m-k}}$, such that $F = F_{i_1} \cap \dots \cap F_{i_{m-k}}$, and furthermore, $\lambda_d(F_{i_1}), \dots, \lambda_d(F_{i_{m-k}})$ determine a subgroup of rank $m - k$ in G_d^m , denoted by G_F . This reconstruction of M tells us that any topological invariant of $\pi_d : M \rightarrow P$ can be determined by (P, λ_d) . Davis and Januszkiewicz showed that $\pi_d : M \rightarrow P$ has a very beautiful algebraic topology in terms of (P, λ_d) . For example, the equivariant cohomology with R_d coefficients of $\pi_d : M \rightarrow P$ is isomorphic to the Stanley–Reisner face ring of P , and the mod 2 Betti numbers (b_0, b_1, \dots, b_m) of M for $d = 1$ and the Betti numbers $(b_0, b_2, \dots, b_{2m})$ of M for $d = 2$ agree with the h -vector (h_0, h_1, \dots, h_m) of P .

In addition, associated with a simple convex m -polytope P with \mathfrak{h} facets $F_1, \dots, F_{\mathfrak{h}}$, Davis and Januszkiewicz also introduced a $G_d^{\mathfrak{h}}$ -manifold $\mathcal{Z}_{P,d}$ of dimension $(d - 1)\mathfrak{h} + m$ over P as follows: first define a map $\theta_d : \{F_1, \dots, F_{\mathfrak{h}}\} \rightarrow R_d^{\mathfrak{h}}$ by mapping $F_i \mapsto e_i$ where $\{e_1, \dots, e_{\mathfrak{h}}\}$ is the standard basis of $R_d^{\mathfrak{h}}$, and then use θ_d to give an equivalence relation \sim_{θ_d} on $P \times G_d^{\mathfrak{h}}$ as in (2.1), so that the required $G_d^{\mathfrak{h}}$ -manifold $\mathcal{Z}_{P,d}$ is just the quotient $P \times G_d^{\mathfrak{h}} / \sim_{\theta_d}$ with a natural $G_d^{\mathfrak{h}}$ -action having orbit space as P . Later on, Buchstaber and Panov [BP] further studied the topology of $\mathcal{Z}_{P,d}$ as a submanifold in the polydisk $(D^d)^{\times \mathfrak{h}}$, and named it a real moment-angle manifold for $d = 1$ and a moment-angle manifold for $d = 2$. Note that a real moment-angle manifold and a moment-angle manifold are often denoted by $\mathbb{R}\mathcal{Z}_P$ and \mathcal{Z}_P , respectively.

As pointed out in [DJ, Nonexample 1.22], given a simple convex m -polytope P with $m > 3$, there may not exist any G_d^m -manifold over P . However, there always exists a (real) moment-angle manifold over P .

When P admits a characteristic function λ_d (so there is a G_d^m -manifold M^{dm} over P reconstructed by (P, λ_d)), regarding $R_d^{\mathfrak{h}}$ as a free module generated by $\{F_1, \dots, F_{\mathfrak{h}}\}$, the map λ_d may linearly extend to a surjection $\widetilde{\lambda}_d : R_d^{\mathfrak{h}} \rightarrow R_d^m$. Then the kernel of $\widetilde{\lambda}_d$ determines a subgroup H of rank $\mathfrak{h} - m$ of $G_d^{\mathfrak{h}}$, which can freely act on $\mathcal{Z}_{P,d}$ such that the quotient manifold $\mathcal{Z}_{P,d}/H$ is exactly equivariantly homeomorphic to the G_d^m -manifold M^{dm} . Thus, the natural projection $\rho_d : \mathcal{Z}_{P,d} \rightarrow M^{dm}$ is a fibration with fiber $G_d^{\mathfrak{h}-m}$. Davis and Januszkiewicz showed in [DJ] that the Borel constructions $EG_d^{\mathfrak{h}} \times_{G_d^{\mathfrak{h}}} \mathcal{Z}_{P,d}$ and $EG_d^m \times_{G_d^m} M^{dm}$ are homotopy-equivalent. For more details of these equivariant manifolds above with many interesting developments and applications, e.g., see [DJ, BP, BBCG, CL, CMS, CPS, IFM, LT, LY, M, MS, U].

2.2. Basic constructions and properties of orbit configuration spaces of G_d^m -manifolds and (real) moment-angle manifolds. Let $\pi_d : M \rightarrow P$ be a dm -dimensional G_d^m -manifold over a simple convex polytope P . Then the product $\pi_d^{\times n} : M^{\times n} \rightarrow P^{\times n}$ is also a dmn -dimensional $(G_d^m)^{\times n}$ -manifold over a simple convex polytope $P^{\times n}$.

Now let $\tilde{\Delta}(P^{\times n})$ be the weak diagonal of $P^{\times n}$, i.e., $\tilde{\Delta}(P^{\times n}) = \bigcup_{1 \leq i < j \leq n} \Delta_{i,j}(P^{\times n})$ where $\Delta_{i,j}(P^{\times n}) = \{(p_1, p_2, \dots, p_n) \in P^{\times n} | p_i = p_j\}$. By definition, we have that $F(P, n) = P^{\times n} - \tilde{\Delta}(P^{\times n})$. By the constructions of M , we see that $F_{G_d^m}(M, n)$ is the pullback from $\pi_d^{\times n} : M^{\times n} \rightarrow P^{\times n}$ via the inclusion $F(P, n) \hookrightarrow P^{\times n}$. So there is the following commutative diagram:

$$\begin{array}{ccc} F_{G_d^m}(M, n) & \longrightarrow & M^{\times n} \\ \pi_d^{\times n} \downarrow & & \downarrow \pi_d^{\times n} \\ F(P, n) & \longrightarrow & P^{\times n}. \end{array}$$

Furthermore, we obtain that $F_{G_d^m}(M, n) \subset M^{\times n}$ is a non-free orbit configuration space, and admits an action of $(G_d^m)^{\times n}$ such that the orbit space is exactly $F(P, n)$.

Proposition 2.1. *Let $\pi_d : M \rightarrow P$ be a dm -dimensional G_d^m -manifold over a simple convex polytope P . Then*

$$F_{G_d^m}(M, n) = \bigcap_{1 \leq i < j \leq n} (M^{\times n} - (\pi_d^{\times n})^{-1}(\Delta_{i,j}(P^{\times n}))).$$

Proof. The required result follows by using the De Morgan formula. \square

Remark 3. In Proposition 2.1, each $M^{\times n} - (\pi_d^{\times n})^{-1}(\Delta_{i,j}(P^{\times n}))$ is homeomorphic to $M^{\times(n-2)} \times F_{G_d^m}(M, 2)$.

Remark 4. Let $M \rightarrow P$ be a quasi-toric manifold over P . As shown in [DJ, Corollary 1.9], there is a conjugation involution τ on M such that its fixed point set M^τ is exactly a small cover over P . This means that there is still an involution on $F_{T^m}(M, n)$ such that its fixed point set is $F_{\mathbb{Z}_2^m}(M^\tau, n)$.

Let $p_d : \mathcal{Z}_{P,d} \rightarrow P$ be the (real) moment-angle manifold over P with \mathfrak{h} facets. Similarly, we see from the constructions of $\mathcal{Z}_{P,d}$ that $F_{G_d^{\mathfrak{h}}}(\mathcal{Z}_{P,d}, n)$ admits an action of $(G_d^{\mathfrak{h}})^{\times n}$ and is the pullback from $p_d^{\times n} : \mathcal{Z}_{P,d}^{\times n} \rightarrow P^{\times n}$ via the inclusion $F(P, n) \hookrightarrow P^{\times n}$, so there is a commutative diagram

$$\begin{array}{ccc} F_{G_d^{\mathfrak{h}}}(\mathcal{Z}_{P,d}, n) & \longrightarrow & \mathcal{Z}_{P,d}^{\times n} \\ p_d^{\times n} \downarrow & & \downarrow p_d^{\times n} \\ F(P, n) & \longrightarrow & P^{\times n}. \end{array}$$

Thus we have that

$$F_{G_d^{\mathfrak{h}}}(\mathcal{Z}_{P,d}, n) = \bigcap_{1 \leq i < j \leq n} (\mathcal{Z}_{P,d}^{\times n} - (p_d^{\times n})^{-1}(\Delta_{i,j}(P^{\times n}))).$$

If we assume that there exists a G_d^m -manifold $\pi_d : M \rightarrow P$ over P , then we know that $\mathcal{Z}_{P,d}$ is a principal $G_d^{\mathfrak{h}-m}$ -bundle over M , denoted by $\rho_d : \mathcal{Z}_{P,d} \rightarrow M$. Then we have that $p_d = \pi_d \circ \rho_d$. Furthermore, we have the following commutative

diagram:

$$\begin{array}{ccc}
F_{G_d^{\mathfrak{h}}}(\mathcal{Z}_{P,d}, n) & \longrightarrow & \mathcal{Z}_{P,d}^{\times n} \\
\bar{\rho}_d^{\times n} \downarrow & & \downarrow \rho_d^{\times n} \\
F_{G_d^m}(M, n) & \longrightarrow & M^{\times n} \\
\bar{\pi}_d^{\times n} \downarrow & & \downarrow \pi_d^{\times n} \\
F(P, n) & \longrightarrow & P^{\times n}.
\end{array}$$

It is not difficult to see that $\bar{\rho}_d^{\times n} : F_{G_d^{\mathfrak{h}}}(\mathcal{Z}_{P,d}, n) \rightarrow F_{G_d^m}(M, n)$ is a fibration with fiber $(G_d^{\mathfrak{h}-m})^{\times n}$. In the same way as in [DJ, 4.1] and [BP, Proposition 6.34], we have the following homotopy-equivalent Borel constructions

$$E(G_d^{\mathfrak{h}})^{\times n} \times_{(G_d^{\mathfrak{h}})^{\times n}} F_{G_d^{\mathfrak{h}}}(\mathcal{Z}_{P,d}, n) \simeq E(G_d^m)^{\times n} \times_{(G_d^m)^{\times n}} F_{G_d^m}(M, n).$$

Thus we conclude that

Proposition 2.2. *Given a simple convex m -polytope P with \mathfrak{h} facets, assume that $\pi_d : M \rightarrow P$ is a G_d^m -manifold over P . Let $p_d : \mathcal{Z}_{P,d} \rightarrow P$ be the (real) moment-angle manifold over P . Then the equivariant cohomologies of $F_{G_d^m}(M, n)$ and $F_{G_d^{\mathfrak{h}}}(\mathcal{Z}_{P,d}, n)$ are isomorphic, i.e.,*

$$H_{(G_d^m)^{\times n}}^*(F_{G_d^m}(M, n)) \cong H_{(G_d^{\mathfrak{h}})^{\times n}}^*(F_{G_d^{\mathfrak{h}}}(\mathcal{Z}_{P,d}, n)).$$

3. EULER CHARACTERISTIC OF $F_{G_d^m}(M, n)$

The objective of this section is to calculate the Euler characteristic $\chi(F_{G_d^m}(M, n))$ for a dm -dimensional G_d^m -manifold $\pi_d : M \rightarrow P$.

3.1. Euler characteristic of union—the inclusion-exclusion principle. Suppose that X_1, \dots, X_N are CW-complexes such that their all possible nonempty intersections are subcomplexes of $X_1 \cup \dots \cup X_N$. Let Δ^N be the abstract simplex on vertex set $[N] = \{1, \dots, N\}$, i.e., $\Delta^N = 2^{[N]}$ (the power set of $[N]$). For each $a \in 2^{[N]}$, set

$$X_a = \begin{cases} \bigcap_{i \in a} X_i & \text{if } a \neq \emptyset \\ \bigcup_{i=1}^N X_i & \text{if } a = \emptyset. \end{cases}$$

Since each pair (X_i, X_j) is an excisive couple of $X_i \cup X_j$, we have the following well-known formula for euler characteristics.

Proposition 3.1 (Inclusion-exclusion principle).

$$\chi(X_{\emptyset}) = \sum_{\substack{a \in 2^{[N]} \\ a \neq \emptyset}} (-1)^{|a|-1} \chi(X_a).$$

3.2. The h -polynomial and the cell-vector of P . Let P be a simple convex m -polytope. The f -vector of P is an integer vector $(f_0, f_1, \dots, f_{m-1})$, where f_i is the number of faces of P of codimension $i+1$ (i.e., of dimension $m-i-1$). Then the h -vector of P is the integer vector (h_0, h_1, \dots, h_m) defined from the following equation

$$(3.1) \quad h_0 + h_1 t + \dots + h_m t^m = (t-1)^m + f_0(t-1)^{m-1} + \dots + f_{m-2}(t-1) + f_{m-1}.$$

The f -vector and the h -vector determine each other by Equation (3.1).

Let $\mathbf{h}_P(t) = h_0 + h_1 t + \cdots + h_m t^m$, which is a polynomial in $\mathbb{Z}[t]$. We call $\mathbf{h}_P(t)$ the *h-polynomial* of P . Given a finite CW-complex X of dimension l , the *cell-vector* $c(X)$ of X is the integer vector (c_0, c_1, \dots, c_l) where c_i denotes the number of all i -cells in X . Each simple convex m -polytope P has a natural cell decomposition such that the interior $\text{int}F$ of an i -face F of P is an i -cell. Thus,

$$c(P) = (f_{m-1}, f_{m-2}, \dots, f_1, f_0, 1)$$

where $f(P) = (f_0, f_1, \dots, f_{m-1})$ is the f -vector of P .

For an arbitrary positive integer ℓ , by $\Delta(P^{\times \ell})$ we denote the strong diagonal of $P^{\times \ell}$, i.e.,

$$\Delta(P^{\times \ell}) = \{(p, p, \dots, p) \in P^{\times \ell} | p \in P\}.$$

Lemma 3.1. *Let $\pi_d : M \rightarrow P$ be a dm -dimensional G_d^m -manifold over a simple convex m -polytope P . Then for a positive integer ℓ , the Euler characteristic of $(\pi_d^{\times \ell})^{-1}(\Delta(P^{\times \ell}))$ is*

$$\chi((\pi_d^{\times \ell})^{-1}(\Delta(P^{\times \ell}))) = \begin{cases} \mathbf{h}_P(1 - 2^\ell) & \text{if } d = 1 \\ \mathbf{h}_P(1) = \chi(M) & \text{if } d = 2. \end{cases}$$

Proof. Fix the cell decomposition of P as above such that its cell-vector $c(P) = (f_{m-1}, f_{m-2}, \dots, f_1, f_0, 1)$. Let F be a face of dimension i . By [DJ, Lemma 1.3], we know that $\pi_d^{-1}(F)$ is still a di -dimensional G_d^i -manifold over F , and in particular, $\pi_d^{-1}(\text{int}F) = G_d^i \times \text{int}F$. When $d = 1$, $\pi_1^{-1}(\text{int}F)$ is the disjoint union of 2^i copies of $\text{int}F$. Since the strong diagonal $\Delta(F^{\times \ell})$ is combinatorially equivalent to F , $(\pi_1^{\times \ell})^{-1}(\Delta((\text{int}F)^{\times \ell}))$ is the disjoint union of $2^{i\ell}$ copies of $\Delta((\text{int}F)^{\times \ell})$. Thus, the cell-vector of $(\pi_1^{\times \ell})^{-1}(\Delta(P^{\times \ell}))$ is

$$(f_{m-1}, 2^\ell f_{m-2}, \dots, 2^{i\ell} f_{m-i-1}, \dots, 2^{(m-2)\ell} f_1, 2^{(m-1)\ell} f_0, 2^{m\ell}).$$

Furthermore, by Equation (3.1)

$$\begin{aligned} & \chi((\pi_1^{\times \ell})^{-1}(\Delta(P^{\times \ell}))) \\ &= f_{m-1} - 2^\ell f_{m-2} + \cdots + (-1)^i 2^{i\ell} f_{m-i-1} + \cdots + (-1)^{m-1} 2^{(m-1)\ell} f_0 + (-1)^m 2^{m\ell} \\ &= \mathbf{h}_P(1 - 2^\ell). \end{aligned}$$

When $d = 2$, we see easily that $(\pi_2^{\times \ell})^{-1}(\Delta((\text{int}F)^{\times \ell})) = T^{i\ell} \times \Delta((\text{int}F)^{\times \ell})$. Now, for $i > 0$, we give a cell decomposition for each circle S^1 in $T^{i\ell}$, with one 0-cell and one 1-cell. Then $(\pi_2^{\times \ell})^{-1}(\Delta((\text{int}F)^{\times \ell}))$ contains $\binom{i\ell}{k}$ cells of dimension- $(i+k)$ for $0 \leq k \leq i\ell$. Thus, all i -cells of P contribute $\binom{i\ell}{k} f_{m-i-1}$ cells of dimension- $(i+k)$ in $(\pi_2^{\times \ell})^{-1}(\Delta(P^{\times \ell}))$ where $0 \leq k \leq i\ell$. Since $\sum_{k=0}^{i\ell} (-1)^k \binom{i\ell}{k} = 0$ for every $i > 0$, by a direct calculation we have

$$\chi((\pi_2^{\times \ell})^{-1}(\Delta(P^{\times \ell}))) = f_{m-1} = \mathbf{h}_P(1) = \chi(M)$$

as desired. \square

3.3. Subgraphs of \mathcal{K}_n and partitions of n and $[n]$. Let \mathcal{K}_n be the complete graph of degree $n-1$, which contains n vertices and $\binom{n}{2}$ edges. We label n vertices of \mathcal{K}_n by $1, \dots, n$ respectively, and $\binom{n}{2}$ edges by pairs (i, j) , $1 \leq i < j \leq n$, respectively. Thus we may identify \mathcal{K}_n with the 1-skeleton of the abstract $(n-1)$ -simplex $\Delta^n = 2^{[n]}$ on vertex set $[n] = \{1, \dots, n\}$.

Definition 3.1. A subgraph Γ of \mathcal{K}_n is said to be *vertex-full* if the vertex set of Γ is $[n]$.

By $\mathbf{VF}(\mathcal{K}_n)$ we denote the set of all vertex-full subgraphs of \mathcal{K}_n .

Lemma 3.2. *There is a one-to-one correspondence between all subsets of the power set $2^{[[n]]}$ and all vertex-full subgraphs of $\mathbf{VF}(\mathcal{K}_n)$, where $[[n]] = \{(i, j) | 1 \leq i < j \leq n\}$.*

Proof. Each vertex-full subgraph Γ of \mathcal{K}_n uniquely determines a subset $E(\Gamma)$ of $2^{[[n]]}$, where $E(\Gamma)$ denotes the set of all edges of Γ . Note that the discrete subgraph $[n]$ of \mathcal{K}_n corresponds to the empty set \emptyset of $2^{[[n]]}$. Conversely, let the empty set \emptyset of $2^{[[n]]}$ correspond to the discrete subgraph $[n]$ of \mathcal{K}_n . Each nonempty subset of $2^{[[n]]}$ determines a unique subgraph Γ of \mathcal{K}_n . If the vertex set of Γ does not cover $[n]$, then we can add those missing vertices as one-point subgraphs to Γ to give the required vertex-full subgraph of \mathcal{K}_n . \square

Given a vertex-full subgraph Γ in $\mathbf{VF}(\mathcal{K}_n)$, define

$$\Delta_\Gamma(P^{\times n}) = \begin{cases} \bigcap_{(i,j) \in E(\Gamma)} \Delta_{i,j}(P^{\times n}) & \text{if } E(\Gamma) \neq \emptyset \\ P^{\times n} & \text{if } E(\Gamma) = \emptyset \end{cases}$$

where $E(\Gamma)$ denotes the set of all edges of Γ . Generally, Γ may not be connected. By $C(\Gamma)$ we denote the set of all connected subgraphs of Γ . Write $C(\Gamma) = \{\Gamma_1, \dots, \Gamma_s\}$. Then $\Gamma = \coprod_{k=1}^s \Gamma_k$ (a disjoint union of $\Gamma_1, \dots, \Gamma_s$).

Lemma 3.3. *Let Γ be a vertex-full subgraph in $\mathbf{VF}(\mathcal{K}_n)$ with $C(\Gamma) = \{\Gamma_1, \dots, \Gamma_s\}$. Then*

$$\Delta_\Gamma(P^{\times n}) = \prod_{k=1}^s \Delta(P^{\times |V(\Gamma_k)|})$$

where $V(\Gamma_k)$ denotes the vertex set of Γ_k .

Proof. Obviously, if $\Gamma = [n]$, then the required equality holds. Suppose that $\Gamma \neq [n]$. For each component Γ_k of Γ , $\bigcap_{(i,j) \in E(\Gamma_k)} \Delta_{i,j}(P^{\times n})$ is combinatorially equivalent to $P^{\times n - |V(\Gamma_k)|} \times \Delta(P^{\times |V(\Gamma_k)|})$. Thus

$$\Delta_\Gamma(P^{\times n}) = \bigcap_{k=1}^s \bigcap_{(i,j) \in E(\Gamma_k)} \Delta_{i,j}(P^{\times n}) = \prod_{k=1}^s \Delta(P^{\times |V(\Gamma_k)|})$$

as desired. \square

Recall that a *partition* of n is an unordered sequence (n_1, \dots, n_s) of positive integers with sum n , and a *partition* of $[n]$ is an unordered sequence of nonempty subsets of $[n]$ which are pairwise disjoint and whose union is $[n]$. Clearly, every vertex-full subgraph $\Gamma = \coprod_{k=1}^s \Gamma_k$ of $\mathbf{VF}(\mathcal{K}_n)$ gives a partition $(|V(\Gamma_1)|, \dots, |V(\Gamma_s)|)$ of n , denoted by $n(\Gamma)$. In addition, each vertex-full subgraph of $\mathbf{VF}(\mathcal{K}_n)$ also determines a partition $(V(\Gamma_1), \dots, V(\Gamma_s))$ of $[n]$.

Lemma 3.4. *Let $I = (n_1, \dots, n_s)$ be a partition of n . Then the number of those combinatorially equivalent vertex-full subgraphs Γ with $n(\Gamma) = I$ of $\mathbf{VF}(\mathcal{K}_n)$ is*

$$\frac{n!}{n_1! \cdots n_s! r_1! \cdots r_s!}$$

where r_i denotes the time number that n_i appears in I .

Proof. Obviously, those combinatorially equivalent vertex-full subgraphs Γ with $n(\Gamma) = I$ of $\mathbf{VF}(\mathcal{K}_n)$ bijectively correspond to those partitions (a_1, \dots, a_s) with $|a_k| = n_k$ of $[n]$. The desired number then follows from an easy argument. \square

3.4. Calculation of Euler characteristic. Now let us calculate $\chi(F_{G_d^m}(M, n))$ for a G_d^m -manifold $\pi_d : M \rightarrow P$ over a simple convex polytope P .

Definition 3.2. Let $I = (n_1, \dots, n_s)$ be a partition of n . Define

$$\mathcal{C}_I = \sum_{\substack{\Gamma \in \mathbf{VF}(\mathcal{K}_n) \\ n(\Gamma) = I}} (-1)^{|E(\Gamma)|}.$$

Theorem 3.1. Let $\pi_d : M \rightarrow P$ be a dm -dimensional G_d^m -manifold over a simple convex polytope. Then

$$\chi(F_{G_d^m}(M, n)) = \begin{cases} (-1)^{mn} \sum_{I=(n_1, \dots, n_s)} \mathcal{C}_I \prod_{k=1}^s \mathbf{h}_P(1 - 2^{n_k}) & \text{if } d = 1 \\ \chi(F(M, n)) & \text{if } d = 2 \end{cases}$$

where $I = (n_1, \dots, n_s)$ runs over all partitions of n .

Proof. First, we calculate $\chi((\pi_d^{\times n})^{-1}(\tilde{\Delta}(P^{\times n})))$ by Proposition 2.1. Since $2^{[n]}$ is combinatorially equivalent to $2^{[N]}$ where $N = \binom{n}{2}$, by Proposition 3.1 and Lemmas 3.1-3.3, we have that

$$\begin{aligned} & \chi((\pi_d^{\times n})^{-1}(\tilde{\Delta}(P^{\times n}))) \\ &= \sum_{\substack{\Gamma = \coprod_{k=1}^s \Gamma_k \in \mathbf{VF}(\mathcal{K}_n) \\ E(\Gamma) \neq \emptyset}} (-1)^{|E(\Gamma)|-1} \chi((\pi_d^{\times n})^{-1}(\Delta_\Gamma(P^{\times n}))) \\ &= \sum_{\substack{\Gamma = \coprod_{k=1}^s \Gamma_k \in \mathbf{VF}(\mathcal{K}_n) \\ E(\Gamma) \neq \emptyset}} (-1)^{|E(\Gamma)|-1} \chi((\pi_d^{\times n})^{-1}(\prod_{k=1}^s \Delta(P^{\times |V(\Gamma_k)|}))) \\ &= \sum_{\substack{\Gamma = \coprod_{k=1}^s \Gamma_k \in \mathbf{VF}(\mathcal{K}_n) \\ E(\Gamma) \neq \emptyset}} (-1)^{|E(\Gamma)|-1} \prod_{k=1}^s \chi((\pi_d^{\times |V(\Gamma_k)|})^{-1}(\Delta(P^{\times |V(\Gamma_k)|}))) \\ &= \begin{cases} \sum_{\substack{\Gamma = \coprod_{k=1}^s \Gamma_k \in \mathbf{VF}(\mathcal{K}_n) \\ E(\Gamma) \neq \emptyset}} (-1)^{|E(\Gamma)|-1} \prod_{k=1}^s \mathbf{h}_P(1 - 2^{|V(\Gamma_k)|}) & \text{if } d = 1 \\ \sum_{\substack{\Gamma = \coprod_{k=1}^s \Gamma_k \in \mathbf{VF}(\mathcal{K}_n) \\ E(\Gamma) \neq \emptyset}} (-1)^{|E(\Gamma)|-1} \chi(M)^s & \text{if } d = 2. \end{cases} \end{aligned}$$

Since each vertex-full subgraph $\Gamma = \coprod_{k=1}^s \Gamma_k$ of \mathcal{K}_n corresponds to a unique partition $n(\Gamma) = (|V(\Gamma_1)|, \dots, |V(\Gamma_s)|)$ of n , we further have that

$$\chi((\pi_d^{\times n})^{-1}(\tilde{\Delta}(P^{\times n}))) = \begin{cases} - \sum_{\substack{I=(n_1, \dots, n_s) \\ I \neq (1, \dots, 1)}} \mathcal{C}_I \prod_{k=1}^s \mathbf{h}_P(1 - 2^{n_k}) & \text{if } d = 1 \\ - \sum_{\substack{I=(n_1, \dots, n_s) \\ I \neq (1, \dots, 1)}} \mathcal{C}_I \chi(M)^s & \text{if } d = 2 \end{cases}$$

where I runs over those partitions except for $(1, \dots, 1)$ of n . A direct calculation gives that $\mathcal{C}_{(1, \dots, 1)} = 1$, so

$$\chi(M^{\times n}) = \begin{cases} \mathcal{C}_{(1, \dots, 1)} \mathbf{h}_P(-1)^n & \text{if } d = 1 \\ \mathcal{C}_{(1, \dots, 1)} \mathbf{h}_P(1)^n = \mathcal{C}_{(1, \dots, 1)} \chi(M)^n & \text{if } d = 2 \end{cases}$$

by Lemma 3.1.

Next, by Lefschetz duality theorem and Proposition 2.1, we conclude that

$$\begin{aligned}
& \chi(F_{G_d^m}(M, n)) \\
&= (-1)^{dmn} (\chi(M^{\times n}) - \chi((\pi_d^{\times n})^{-1}(\tilde{\Delta}(P^{\times n}))) \\
&= \begin{cases} (-1)^{mn} (\chi(M^{\times n}) + \sum_{\substack{I=(n_1, \dots, n_s) \\ I \neq (1, \dots, 1)}} \mathcal{C}_I \prod_{k=1}^s \mathbf{h}_P(1 - 2^{n_k})) & \text{if } d = 1 \\ \chi(M^{\times n}) + \sum_{\substack{I=(n_1, \dots, n_s) \\ I \neq (1, \dots, 1)}} \mathcal{C}_I \chi(M)^s & \text{if } d = 2 \end{cases} \\
&= \begin{cases} (-1)^{mn} \sum_{I=(n_1, \dots, n_s)} \mathcal{C}_I \prod_{k=1}^s \mathbf{h}_P(1 - 2^{n_k}) & \text{if } d = 1 \\ \sum_{I=(n_1, \dots, n_s)} \mathcal{C}_I \chi(M)^s & \text{if } d = 2. \end{cases}
\end{aligned}$$

When $d = 2$, in a similar way as above, we have that

$$\begin{aligned}
\chi(F(M, n)) &= (-1)^{2mn} (\chi(M^{\times n}) - \chi(\tilde{\Delta}(M^{\times n}))) \\
&= \chi(M^{\times n}) - \chi(\tilde{\Delta}(M^{\times n})) \\
&= \sum_{I=(n_1, \dots, n_s)} \mathcal{C}_I \chi(M)^s
\end{aligned}$$

as desired. \square

We know from [Mu] that the Lefschetz duality theorem holds for compact triangulated homology manifolds. Thus, using the proof method of Theorem 3.1, we can obtain the following formula for more general non-equivariant configuration spaces.

Theorem 3.2. *Let M be a compact triangulated homology m -manifold. Then*

$$\chi(F(M, n)) = (-1)^{mn} \sum_I \mathcal{C}_I \chi(M)^s$$

where $I = (n_1, \dots, n_s)$ runs over all partitions of n .

In Theorem 3.2, if we further write

$$\chi(F(M, n)) = (-1)^{nm} \sum_I \mathcal{C}_I \chi(M)^s = (-1)^{mn} \sum_{s=1}^n \left(\sum_{I=(n_1, \dots, n_s)} \mathcal{C}_I \right) \chi(M)^s$$

then we see that $\chi(F(M, n))$ is actually a polynomial (with \mathbb{Z} coefficients) of $\chi(M)$ of degree n . By $g(t)$ we denote this polynomial in $\mathbb{Z}[t]$. We first complete the proof of Theorem 1.2.

Proof of Theorem 1.2. It suffices to show that $g(t) = (-1)^{mn} \prod_{k=0}^{n-1} (t - k)$. For $0 \leq k < n$, choose M as a set consisting of k points. Then M is a 0-dimensional manifold if $0 < k < n$, and a empty set (or -1 -dimensional manifold) if $k = 0$. Thus

$$\chi(M) = \begin{cases} k & \text{if } 0 < k < n \\ 0 & \text{if } k = 0. \end{cases}$$

By using the pigeonhole principle, since $k < n$, we see that

$$F(M, n) = \{(x_1, \dots, x_n) \in M^{\times n} \mid x_i \neq x_j \text{ for } i \neq j\}$$

must be empty, so $\chi(F(M, n)) = 0$. This implies that $g(k) = 0$ for $0 \leq k < n$, and thus each k is a root of $g(t)$. Furthermore, we can write $g(t) = (-1)^{mn} c \prod_{k=0}^{n-1} (t - k)$

where c is a constant number. Since $\mathcal{C}_{(1,\dots,1)} = 1$, we conclude that c must be 1. This completes the proof. \square

Corollary 3.3.

$$\mathcal{C}_{(n)} = (-1)^{n-1}(n-1)!.$$

Proof. This can be obtained by comparing the coefficients of $\chi(M)$ on both sides of the following equality

$$\sum_{s=1}^n \left(\sum_{I=(n_1,\dots,n_s)} C_I \right) \chi(M)^s = \prod_{k=0}^{n-1} (\chi(M) - k).$$

\square

Finally, to complete the proof of Theorem 1.1, it remains to determine the number \mathcal{C}_I for every partition I of n .

Proposition 3.2. *Let $I = (n_1, \dots, n_s)$ be a partition of n . Then*

$$\mathcal{C}_I = \frac{n!(-1)^{n-s}}{r_1!r_2!\cdots r_s!n_1n_2\cdots n_s}$$

where r_k denotes the time number that n_k appears in I .

Proof. Let \mathcal{A}_I denote the set of those vertex-full subgraphs Γ with $n(\Gamma) = I$ of $\mathbf{VF}(\mathcal{K}_n)$, all of which are not combinatorially equivalent to each other. By Lemma 3.4 and Corollary 3.3, we have that

$$\begin{aligned} \mathcal{C}_I &= \sum_{\substack{\Gamma \in \mathbf{VF}(\mathcal{K}_n) \\ n(\Gamma) = I}} (-1)^{|E(\Gamma)|} \\ &= \frac{n!}{n_1!\cdots n_s!r_1!\cdots r_s!} \sum_{\substack{\Gamma = \coprod_{k=1}^s \Gamma_k \in \mathcal{A}_I \\ |V(\Gamma_k)| = n_k}} \prod_{k=1}^s (-1)^{|E(\Gamma_k)|} \\ &= \frac{n!}{n_1!\cdots n_s!r_1!\cdots r_s!} \prod_{k=1}^s \sum_{\Gamma_k} (-1)^{|E(\Gamma_k)|} \\ &= \frac{n!}{n_1!\cdots n_s!r_1!\cdots r_s!} \prod_{k=1}^s \mathcal{C}_{(n_k)} \\ &= \frac{n!}{n_1!\cdots n_s!r_1!\cdots r_s!} \prod_{k=1}^s (-1)^{n_k-1}(n_k-1)! \\ &= \frac{n!(-1)^{n-s}}{r_1!r_2!\cdots r_s!n_1n_2\cdots n_s} \end{aligned}$$

as desired. \square

Example 3.1. By the formula of Theorem 1.1, we have that

$$\chi(F_{\mathbb{Z}_2^m}(M, 2)) = \mathbf{h}_P(-1)^2 - \mathbf{h}_P(-3)$$

and

$$\chi(F_{\mathbb{Z}_2^m}(M, 3)) = (-1)^{3m}(\mathbf{h}_P(-1)^3 - 3\mathbf{h}_P(-1)\mathbf{h}_P(-3) + 2\mathbf{h}_P(-7))$$

where M is a small cover over P .

Corollary 3.4. *Given a simple convex m -polytope P with \mathfrak{h} facets, assume that there exists a small cover over P . Then*

$$\chi(F_{\mathbb{Z}_2^{\mathfrak{h}}}(\mathcal{Z}_{P,1}, n)) = (-1)^{mn} 2^{(h-m)n} \sum_{I=(n_1, \dots, n_s)} \frac{n!(-1)^{n-s}}{r_1! r_2! \cdots r_s! n_1 n_2 \cdots n_s} \prod_{k=1}^s \mathbf{h}_P(1-2^{n_k})$$

where $I = (n_1, \dots, n_s)$ runs over all partitions of n , and r_k is the time number that n_k appears in I .

Proof. Let M be a small cover over P . Then $F_{\mathbb{Z}_2^{\mathfrak{h}}}(\mathcal{Z}_{P,1}, n)$ is a principal $(\mathbb{Z}_2^{\mathfrak{h}-m})^{\times n}$ -bundle over $F_{\mathbb{Z}_2^m}(M, n)$. By [AP, p. 86, (1.5)(d)] we have that

$$\chi(F_{\mathbb{Z}_2^{\mathfrak{h}}}(\mathcal{Z}_{P,1}, n)) = |(\mathbb{Z}_2^{\mathfrak{h}-m})^{\times n}| \chi(F_{\mathbb{Z}_2^m}(M, n)).$$

Moreover, the required result follows from Theorem 1.1. \square

Remark 5. It should be interesting to give an explicit formula of $\chi(F_{\mathbb{Z}_2^{\mathfrak{h}}}(\mathcal{Z}_{P,1}, n))$ without the existence assumption of a small cover over P in Corollary 3.4.

Proposition 3.3. *Let P be a simple convex polytope with \mathfrak{h} facets. Then*

$$\chi(F_{T^{\mathfrak{h}}}(\mathcal{Z}_{P,2}, n)) = 0.$$

Proof. We know from [BP, Proposition 7.29] that the diagonal circle subgroup of $T^{\mathfrak{h}}$ acts freely on $\mathcal{Z}_{P,2}$, so the diagonal circle subgroup of $(T^{\mathfrak{h}})^{\times n}$ also acts freely on $F_{T^{\mathfrak{h}}}(\mathcal{Z}_{P,2}, n)$. Therefore, $F_{T^{\mathfrak{h}}}(\mathcal{Z}_{P,2}, n)$ admits a principal S^1 -bundle structure, which induces that $\chi(F_{T^{\mathfrak{h}}}(\mathcal{Z}_{P,2}, n)) = 0$ by [AP, p. 86, (1.5)(c)]. \square

Remark 6. Buchstaber and Panov [BP] expanded the construction of $\mathcal{Z}_{P,d}$ over a simple convex polytope P to the case of general simplicial complex K . The resulting space denoted by $\mathcal{Z}_{K,d}$ is not a manifold in general, called the (real) moment-angle complex. When $d = 2$, $\mathcal{Z}_{K,2}$ still admits a principal S^1 -bundle structure, so the Euler characteristic of its orbit configuration space is zero. It should be also interesting to give an explicit formula for the Euler characteristic of the orbit configuration space of $\mathcal{Z}_{K,1}$ in terms of the combinatorial data of K . In addition, it was showed in [CL, U] that the Halperin-Carlsson conjecture holds for $\mathcal{Z}_{K,d}$ with the restriction free action. Naturally, we wish to know whether this is also true for the orbit configuration space of $\mathcal{Z}_{K,d}$.

4. HOMOTOPY TYPE OF $F_{G_d^m}(M, n)$ FOR $m = 1$ OR $n = 2$

Throughout the following, assume that $\pi_d : M \rightarrow P$ is a dm -dimensional G_d^m -manifold over a simple convex m -polytope P . It is easy to see that $F(P, n)$ is disconnected if $m = 1$ and path-connected if $m > 1$ since $\Delta(P^{\times 2})$ is combinatorially equivalent to P .

4.1. Homotopy type of $F_{G_d^1}(M, n)$.

Theorem 4.1. *Let $\pi_d : M \rightarrow P$ be a d -dimensional G_d^1 -manifold over P . Then, when $d = 1$, $F_{\mathbb{Z}_2}(M, n)$ has the same homotopy type as $n!2^{n-2}$ points, and when $d = 2$, $F_{S^1}(M, n)$ has the same homotopy type as a disjoint union of $n!$ copies of T^{n-2} .*

Proof. It is well-known that when $d = 1$, M is a circle S^1 with a reflection fixing two isolated points such that the orbit polytope P is a 1-dimensional simplex, and when $d = 2$, M is a 2-sphere S^2 with a rotation action of S^1 , fixing two isolated points, such that the orbit polytope P is also a 1-dimensional simplex. Since a 1-simplex is homeomorphic to the interval $[0, 1]$, we may identify $P^{\times n}$ with the cube $[0, 1]^{\times n}$. It is easy to see that each point $x = (x_1, \dots, x_n) \in F(P, n) \subset [0, 1]^{\times n}$ determines a unique permutation $(\sigma(1), \dots, \sigma(n))$ of $[n]$ such that $x_i < x_j$ as long as $\sigma(i) < \sigma(j)$, where $\sigma \in \mathbf{S}_n$, and \mathbf{S}_n is the symmetric group on $[n]$. Define a homotopy $H : F(P, n) \times [0, 1] \rightarrow F(P, n)$ by

$$((x_1, \dots, x_n), t) \mapsto ((1-t)x_1 + t\frac{\sigma(1)-1}{n-1}, \dots, (1-t)x_n + t\frac{\sigma(n)-1}{n-1}).$$

An easy argument shows that this homotopy H is a deformation retraction of $F(P, n)$ onto $n!$ points in $\mathcal{A} = \{(\frac{\sigma(1)-1}{n-1}, \dots, \frac{\sigma(n)-1}{n-1}) \mid \sigma \in \mathbf{S}_n\} \subset F(P, n)$. This means that $F(P, n)$ contains $n!$ connected components $C_\sigma, \sigma \in \mathbf{S}_n$, each of which may continuously collapse to a point in \mathcal{A} . For each σ , since $0, 1 \in \{\frac{\sigma(1)-1}{n-1}, \dots, \frac{\sigma(n)-1}{n-1}\}$ and $\{\frac{\sigma(1)-1}{n-1}, \dots, \frac{\sigma(n)-1}{n-1}\} - \{0, 1\}$ is in the open interval $(0, 1)$, C_σ has also the deformation retract R_σ that is homeomorphic to an $(n-2)$ -dimensional open ball B_σ in $F(P, n)$, and is contained in $\partial P^{\times n}$. Therefore, each R_σ can be chosen in the interior of an $(n-2)$ -face in $P^{\times n}$, so by [DJ, Lemma 4.1],

$$(\pi_d^{\times n})^{-1}(R_\sigma) = G_d^{n-2} \times R_\sigma = \begin{cases} \mathbb{Z}_2^{n-2} \times R_\sigma & \text{if } d = 1 \\ T^{n-2} \times R_\sigma & \text{if } d = 2 \end{cases}$$

which is homotopic to 2^{n-2} points if $d = 1$ and T^{n-2} if $d = 2$. This completes the proof. \square

4.2. An equivariant strong deformation retract of $F_{G_d^m}(M, 2)$ with $m > 1$.
Let $\mathcal{F}(P)$ denote the set of all faces of P .

Lemma 4.1. *There is a strong deformation retraction $H : F(P, 2) \times [0, 1] \rightarrow F(P, 2)$ of $F(P, 2)$ onto*

$$\mathcal{A}(P, 2) = \bigcup_{\substack{F_1, F_2 \in \mathcal{F}(P) \\ F_1 \cap F_2 = \emptyset}} F_1 \times F_2.$$

Proof. First, we note that $F(P, 2) = P \times P - \Delta(P \times P)$ is path-connected since $m > 1$. Since $\Delta(P \times P)$ always contains the interior points of $P \times P$, we have that $F(P, 2)$ can continuously collapse onto $\partial(P \times P) - \Delta(P \times P)$. Since $\partial(P \times P) = \partial P \times P \cup P \times \partial P$, in a similar way, it is easy to see that both $\partial P \times P$ and $P \times \partial P$ can further continuously collapse onto $\partial P \times \partial P - \Delta(\partial P \times \partial P)$. Now we see that $\partial P \times \partial P - \Delta(\partial P \times \partial P)$ is the union of subsets of the following forms

$$F \times F - \Delta(F \times F), F \times F' - \Delta(\partial P \times \partial P), F \times F''$$

where F, F', F'' are facets of P with $F \cap F' \neq \emptyset$ and $F \cap F'' = \emptyset$. In particular, we further see that $F \times F' - \Delta(\partial P \times \partial P)$ can be continuously shrunk to the union of subsets of the following forms

$$(F \cap F') \times (F \cap F') - \Delta((F \cap F')^{\times 2}), Q \times (F \cap F') - \Delta(\partial P \times \partial P), \\ (F \cap F') \times Q' - \Delta(\partial P \times \partial P), Q_1 \times Q'_1$$

where Q and Q_1 are facets of F , Q' and Q'_1 are facets of F' , such that $Q \times (F \cap F') \neq \emptyset$, $(F \cap F') \times Q' \neq \emptyset$, and $Q_1 \cap Q'_1 = \emptyset$. We continuous the above process to

$F \times F - \Delta(F \times F)$, $(F \cap F') \times (F \cap F') - \Delta((F \cap F')^{\times 2})$, $Q \times (F \cap F') - \Delta(\partial P \times \partial P)$, $(F \cap F') \times Q' - \Delta(\partial P \times \partial P)$, and further repeat it whenever possible. This process must end after a finite number of steps, giving finally that $F(P, 2)$ is homotopic to the union $\bigcup_{\substack{F_1 \times F_2 \in \mathcal{F}(P) \\ F_1 \cap F_2 = \emptyset}} F_1 \times F_2$. In particular, it is not difficult to see that this shrinking of $F(P, 2)$ leaves all points of $\bigcup_{\substack{F_1 \times F_2 \in \mathcal{F}(P) \\ F_1 \cap F_2 = \emptyset}} F_1 \times F_2$ fix. This means that there is a strong deformation retraction $H : F(P, 2) \times [0, 1] \rightarrow F(P, 2)$ of $F(P, 2)$ onto $\bigcup_{\substack{F_1 \times F_2 \in \mathcal{F}(P) \\ F_1 \cap F_2 = \emptyset}} F_1 \times F_2$. \square

Remark 7. It is well-known that $F(P, 2)$ has the homotopy type of S^{m-1} , so $\mathcal{A}(P, 2)$ has also the same homotopy type as S^{m-1} .

Theorem 4.2. *There is an equivariant strong deformation retraction of $F_{G_d^m}(M, 2)$ onto*

$$X_d(M, 2) = \bigcup_{\substack{F_1, F_2 \in \mathcal{F}(P) \\ F_1 \cap F_2 = \emptyset}} (\pi_d^{-1})^{\times 2}(F_1 \times F_2).$$

Proof. Since $M = P \times G_d^m / \sim_{\lambda_d}$, we have that $F_{G_d^m}(M, 2) = F(P, 2) \times G_d^{2m} / \sim_{\lambda_d \times \lambda_d}$. For the strong deformation retraction $H : F(P, 2) \times [0, 1] \rightarrow F(P, 2)$ in Lemma 4.1, it may naturally be lift to a strong deformation retraction

$$H_1 : F(P, 2) \times G_d^{2m} \times [0, 1] \rightarrow F(P, 2) \times G_d^{2m}$$

by mapping (a, g, t) to $(H(a, t), g)$. Now assume that two points (a, g) and (a, g') of $F(P, 2) \times G_d^{2m}$ satisfy $(a, g) \sim_{\lambda_d \times \lambda_d} (a, g')$.

Claim A. $H_1(a, g, t) = (H(a, t), g) \sim_{\lambda_d \times \lambda_d} (H(a, t), g') = H_1(a, g', t)$.

If $a \in \text{int}(P^{\times 2})$, then, by the construction of M , $g = g'$. It follows that $H_1(a, g, t) = H_1(a, g', t)$, so $(H(a, t), g) \sim_{\lambda_d \times \lambda_d} (H(a, t), g')$ regardless of whether $H(a, t)$ belongs to $\text{int}(P^{\times 2})$ or not.

If $a \in \partial(P^{\times 2})$, then a belongs to $F \times P$ or $P \times F'$ where F and F' are facets of P . Without the loss of generality, we merely consider the case of $a \in F \times P$ in the following argument. By Lemma 4.1, we see that $\sigma(t) = H(a, t)$ is a path from $H(a, 0)$ to $H(a, 1)$, and there exists a sequence of faces in $F \times P$

$$F \times P \supseteq Q_1 \times Q'_1 \supset \cdots \supset Q_{l-1} \times Q'_{l-1} \supset Q_l \times Q'_l$$

with $Q_i \cap Q'_i \neq \emptyset$ for $i = 1, \dots, l-1$ and $Q_l \cap Q'_l = \emptyset$, such that $\sigma(t)$ continuously runs from $\sigma(0) = H(a, 0) = a \in \text{int}(Q_1 \times Q'_1)$ to $\sigma(1) = H(a, 1) \in \text{int}(Q_l \times Q'_l)$ through

$$\text{int}(Q_1 \times Q'_1) \supset \cdots \supset \text{int}(Q_{l-1} \times Q'_{l-1}) \supset \text{int}(Q_l \times Q'_l).$$

Thus, by the definition of \sim_{λ_d} , we have that $g^{-1}g' \in G_{Q_1} \times G_{Q'_1}$. On the other hand, by the construction of M , we have the following sequence of subgroups of G_d^{2m}

$$G_{Q_1} \times G_{Q'_1} < \cdots < G_{Q_{l-1}} \times G_{Q'_{l-1}} < G_{Q_l} \times G_{Q'_l}$$

where G_Q is the subgroup of G_d^m , determined by Q and the characteristic function of P (see subsection 2.1). This means that $g^{-1}g' \in G_{Q_i \times Q'_i}$ for all $1 \leq i \leq l$. Thus, $(H(a, t), g) \sim_{\lambda_d \times \lambda_d} (H(a, t), g')$ by the definition of \sim_{λ_d} .

Moreover, we conclude by Claim A that H_1 descends to an equivariant strong deformation retraction of $F_{G_d^m}(M, 2)$ onto $X_d(M, 2)$. \square

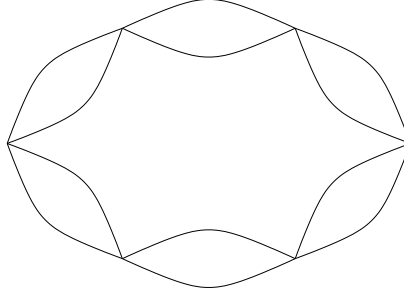
Remark 8. By the intersection property of all $(\pi_d^{-1})^{\times 2}(F_1 \times F_2)$, $F_1, F_2 \in \mathcal{F}(P)$ with $F_1 \cap F_2 = \emptyset$, in the way as shown in section 7, $X_d(M, 2)$ can determine a simplicial complex K such that K is exactly the dual cell decomposition of $\mathcal{A}(P, 2)$.

Corollary 4.3. *Let $\pi_2 : M \longrightarrow P$ be a $2m$ -dimensional quasi-toric manifold over a simple convex polytope P . Then $F_{T^m}(M, 2)$ is simply connected.*

Proof. We know from [DJ, 1.10 and Theorem 3.1] that for any two faces F_1 and F_2 of P , $(\pi_2^{-1})^{\times 2}(F_1 \times F_2)$ is a quasi-toric manifold over $F_1 \times F_2$, and each quasi-toric manifold has no odd-dimensional cells. Thus, $X_2(M, 2)$ is simply connected, so is $F_{T^m}(M, 2)$ by Theorem 4.2. \square

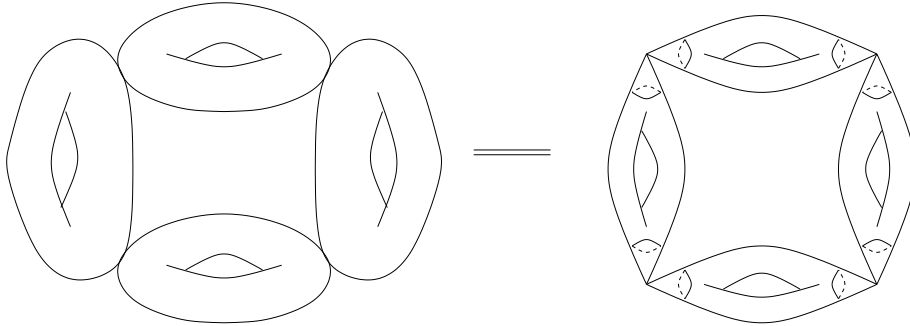
4.3. Examples. Now let us look at the case $m = 2$. In this case, P is a polygon.

- (1) When P is a 3-polygon, we have that $F_{\mathbb{Z}_2^2}(M, 2)$ has the homotopy type of the following 1-dimensional simplicial complex



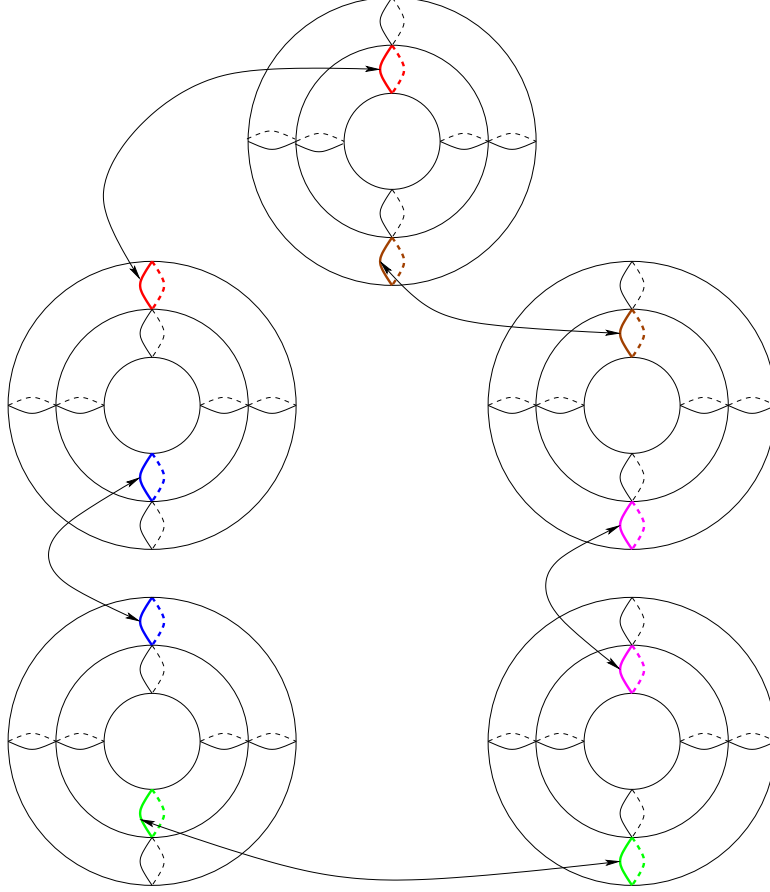
and $F_{T^2}(M, 2)$ has the homotopy type of a 2-dimensional simplicial complex produced by replacing six circles of the above complex by six 2-spheres.

- (2) When P is a 4-polygon, we have that $F_{\mathbb{Z}_2^2}(M, 2)$ has the homotopy type of the following 2-dimensional simplicial complex



and $F_{T^2}(M, 2)$ has the homotopy type of a 4-dimensional simplicial complex produced by replacing four tori of the above complex by four copies of $S^2 \times S^2$.

- (3) When P is a 5-polygon, we have that $F_{\mathbb{Z}_2^2}(M, 2)$ has the homotopy type of



The resulting space is obtained by gluing same colored circles together

and $F_{T^2}(M, 2)$ has the homotopy type of a 4-dimensional simplicial complex produced by replacing all tori and circles of the above complex by $S^2 \times S^2$ and S^2 respectively.

5. BETTI NUMBERS AND (EQUIVARIANT) COHOMOLOGY OF $F_{G_d^m}(M, 2)$

Throughout the following, assume that $\pi_d : M \rightarrow P$ is a dm -dimensional G_d^m -manifold over a simple convex polytope P . By Theorem 4.2, let $\mathcal{H} : F_{G_d^m}(F, 2) \times [0, 1] \rightarrow F_{G_d^m}(F, 2)$ be the equivariant strong deformation retraction of $F_{G_d^m}(F, 2)$ onto $X_d(M, 2)$.

Lemma 5.1. *The equivariant cohomologies of $F_{G_d^m}(F, 2)$ onto $X_d(M, 2)$ are isomorphic, i.e.,*

$$H_{G_d^{2m}}^*(F_{G_d^m}(F, 2)) \cong H_{G_d^{2m}}^*(X_d(M, 2)).$$

Proof. Consider the following equivariant lifting of \mathcal{H}

$$\tilde{\mathcal{H}} : EG_d^{2m} \times F_{G_d^m}(F, 2) \times [0, 1] \rightarrow EG_d^{2m} \times F_{G_d^m}(F, 2)$$

by mapping (x, y, t) to $(x, \mathcal{H}(y, t))$. This lifting $\tilde{\mathcal{H}}$ descends to a deformation retraction

$$EG_d^{2m} \times_{G_d^{2m}} F_{G_d^m}(F, 2) \times [0, 1] \longrightarrow EG_d^{2m} \times_{G_d^{2m}} F_{G_d^m}(F, 2)$$

of $EG_d^{2m} \times_{G_d^{2m}} F_{G_d^m}(F, 2)$ onto $EG_d^{2m} \times_{G_d^{2m}} X_d(M, 2)$, which induces the required result. \square

Now let us look at the cell structure of $X_d(M, 2)$.

Lemma 5.2. *$X_d(M, 2)$ has a perfect cell structure with respect to R_d coefficients in the sense of Morse theory, i.e., the closure of each cell of $X_d(M, 2)$ is a (pseudo) manifold, where R_d is \mathbb{Z}_2 if $d = 1$, and \mathbb{Z} if $d = 2$.*

Proof. As shown in [DJ, Theorem 3.1], each quasi-toric manifold (or small cover) has a perfect cell structure with respect to \mathbb{Z} (or \mathbb{Z}_2) coefficients in the sense of Morse theory. From the construction of $X_d(M, 2)$ in Theorem 4.2, we see that all possible intersections of submanifolds of $\{(\pi_d^{-1})^{\times 2}(F_1 \times F_2) | F_1, F_2 \in \mathcal{F}(P) \text{ with } F_1 \cap F_2 = \emptyset\}$ in $X_d(M, 2)$ are also small covers if $d = 1$ and quasi-toric manifolds if $d = 2$, so the lemma follows from this. \square

Remark 9. Lemma 5.2 implies that \mathbb{Z}_2^{2m} acts trivially on $H^*(F_{\mathbb{Z}_2^m}(M, 2); \mathbb{Z}_2)$.

5.1. The homology of $F_{T^m}(M, 2)$. Now let us observe the homology of $F_{T^m}(M, 2)$ for a $2m$ -dimensional quasi-toric manifold $\pi_2 : M \rightarrow P$.

Proposition 5.1. *Let $\pi_2 : M \rightarrow P$ be a $2m$ -dimensional quasi-toric manifold over a simple convex polytope P . Then the homology of $F_{T^m}(M, 2)$ vanishes in odd dimensions and is free abelian in even dimensions.*

Proof. By Theorem 4.2, it suffices to consider the homology of $X_2(M, 2)$. [DJ, Theorem 3.1] tells us that all cells of each quasi-toric manifold are of even dimension, so $X_2(M, 2)$ has only even-dimensional cells. Then the required result follows from Lemma 5.2. \square

We see that the dual cell decomposition K of $\mathcal{A}(P, 2)$ as a polyhedron is a simplicial complex, and it indicates the intersection property of submanifolds of $\{(\pi_d^{-1})^{\times 2}(F_1 \times F_2) | F_1, F_2 \in \mathcal{F}(P) \text{ with } F_1 \cap F_2 = \emptyset\}$ in $X_d(M, 2)$ for a G_d^m -manifold $\pi_d : M \rightarrow P$. Then we know from section 7 that $X_d(M, 2)$ with K together can be associated to the Mayer–Vietoris spectral sequence $E_{p,q}^1(K), \dots, E_{p,q}^\infty(K)$. The following result is a consequence of Proposition 5.1 and Theorem 7.1.

Corollary 5.1. *Let $\pi_2 : M \rightarrow P$ be a $2m$ -dimensional quasi-toric manifold over a simple convex polytope P . Then the associated Mayer–Vietoris spectral sequence of $X_2(M, 2)$ collapses at the $E_{p,q}^2$ term. Moreover,*

$$H_i(F_{T^m}(M, 2)) = \bigoplus_{p+q=i} E_{p,q}^2(K).$$

5.2. Relation between Betti numbers of $F_{T^m}(M, 2)$ and mod 2 Betti numbers of $F_{\mathbb{Z}_2^m}(M, 2)$. We know from [DJ, Theorem 3.1] that the Betti numbers (resp. mod 2 Betti numbers) of a quasi-toric manifold (resp. a small cover) $M \rightarrow P$ only depends upon the combinatorics (more precisely, the h -vector) of P . Lemma 5.2 tells us that $X_d(P, 2)$ has still a perfect cell structure in the sense of Morse theory, and in particular, this perfect cell structure only depends upon the structure of

$\mathcal{A}(P, 2) = \bigcup_{\substack{F_1, F_2 \in \mathcal{F}(P) \\ F_1 \cap F_2 = \emptyset}} F_1 \times F_2$. In other words, whichever $d = 1$ or 2 , the number of all cells at any dimension in the perfect cell decomposition of $X_d(M, 2)$ is completely determined by the h -vectors of those polytopes $F_1 \times F_2$ with $F_1 \cap F_2 = \emptyset$, $F_1, F_2 \in \mathcal{F}(P)$. Therefore, with Lemma 5.2 and Proposition 5.1 together, we conclude that

Theorem 5.2. *Given a simple convex polytope P , assume that $\pi_d : M \rightarrow P$ is a dm -dimensional G_d^m -manifold over P . Then*

$$b_{2i}(F_{T^m}(M, 2)) = b_i^{\mathbb{Z}_2}(F_{\mathbb{Z}_2^m}(M, 2))$$

and both only depend upon the combinatorial structure of P .

5.3. Equivariant cohomology of $F_{G_d^m}(M, 2)$. Let $\phi_d : EG_d^{2m} \times_{G_d^{2m}} F_{G_d^m}(M, 2) \rightarrow BG_d^{2m}$ be the fibration with fiber $F_{G_d^m}(M, 2)$.

Theorem 5.3. *Let $\pi_d : M \rightarrow P$ be a dm -dimensional G_d^m -manifold over a simple convex polytope P . Then the Leray–Serre spectral sequence (with R_d coefficients) of the fibration ϕ_d collapses at the E_2 term (i.e., $E_\infty^{p,q} = E_2^{p,q}$) if $d = 2$, and has the property $E_2^{p,q} = H^p(B\mathbb{Z}_2^{2m}; \mathbb{Z}_2) \otimes H^q(F_{\mathbb{Z}_2^m}(M, 2); \mathbb{Z}_2)$ if $d = 1$.*

Proof. We see from the proof of Lemma 5.1 that the fibration ϕ_d is homotopic to the fibration $\hat{\phi}_d : EG_d^{2m} \times_{G_d^{2m}} X_d(M, 2) \rightarrow BG_d^{2m}$ with fiber $X_d(M, 2)$, so it suffices to consider the fibration $\hat{\phi}_d$. When $d = 2$, all the differentials in the spectral sequence are trivial since both BT^{2m} and $X_2(M, 2)$ have only even-dimensional cells. Thus, in this case, $E_\infty^{p,q} = E_2^{p,q}$. When $d = 1$, we have that the fundamental group $\pi_1(B\mathbb{Z}_2^{2m}) \cong \mathbb{Z}_2^{2m}$, which acts trivially on $H^*(X_1(M, 2); \mathbb{Z}_2)$ by Remark 9. Thus we have that $E_2^{p,q} = H^p(B\mathbb{Z}_2^{2m}; \mathbb{Z}_2) \otimes H^q(F_{\mathbb{Z}_2^m}(M, 2); \mathbb{Z}_2)$. \square

Corollary 5.4. *Let $\pi_2 : M \rightarrow P$ be a $2m$ -dimensional quasi-toric manifold over a simple convex polytope P . Then $H_{T^{2m}}^*(F_{T^m}(M, 2)) = H^*(F_{T^m}(M, 2)) \otimes H^*(BT^{2m})$ is a free $H^*(BT^{2m})$ -module, and the inclusion of fiber $F_{T^m}(M, 2) \hookrightarrow ET^{2m} \times_{T^{2m}} F_{T^m}(M, 2)$ induces an epimorphism.*

6. CALCULATION OF THE (MOD 2) HOMOLOGY OF $F_{G_d^2}(M, 2)$ AND $F_{G_d^m}(M, 2)$ FOR P TO BE AN m -SIMPLEX

In this section, using Theorem 4.2 and the Mayer–Vietoris spectral sequence we calculate the (mod 2) homology of $F_{G_d^2}(M, 2)$ and $F_{G_d^m}(M, 2)$ for P to be an m -simplex. By Theorem 1.5 this is equivalent to determining the (mod 2) Betti numbers. Our results are stated as follows:

Proposition 6.1. *Let $\pi_d : M \rightarrow P$ be a $2d$ -dimensional G_d^2 -manifold over a polygon P with ℓ vertices. When $\ell = 3$, all nonzero Betti numbers of $F_{\mathbb{Z}_2^2}(M, 2)$ (resp. $F_{T^2}(M, 2)$) are $(b_0, b_1) = (1, 7)$ (resp. $(b_0, b_2) = (1, 7)$); when $\ell > 3$, all nonzero Betti numbers of $F_{\mathbb{Z}_2^2}(M, 2)$ (resp. $F_{T^2}(M, 2)$) are $(b_0, b_1, b_2) = (1, 2\ell + 1, \ell(\ell - 3))$ (resp. $(b_0, b_2, b_4) = (1, 2\ell + 1, \ell(\ell - 3))$). In particular, the non-vanishing homology of $F_{\mathbb{Z}_2^2}(M, 2)$ is free abelian.*

Remark 10. As we have seen in Proposition 6.1, we actually determine the integral homology of $F_{\mathbb{Z}_2^2}(M, 2)$. However, unlike 2-dimensional small covers, the non-vanishing homology of $F_{\mathbb{Z}_2^2}(M, 2)$ has no torsion. This can also be seen from the special examples in subsection 4.3.

Proposition 6.2. *Let $\pi_d : M \rightarrow \Delta^m$ be a dm -dimensional G_d^m -manifold over an m -simplex Δ^m . When $d = 1$, all nonzero mod 2 Betti numbers of $F_{\mathbb{Z}_2^m}(M, 2)$ are*

$$(b_0^{\mathbb{Z}_2}, b_1^{\mathbb{Z}_2}, \dots, b_{m-2}^{\mathbb{Z}_2}, b_{m-1}^{\mathbb{Z}_2}) = (1, 2, \dots, m-1, \frac{3^{m+1} + 2m - 3}{4}).$$

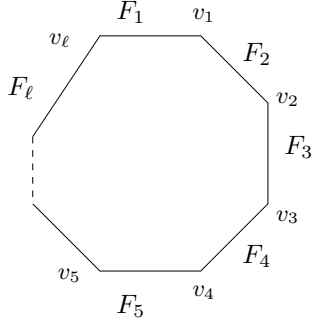
When $d = 2$, all nonzero Betti numbers of $F_{T^m}(M, 2)$ are

$$(b_0, b_2, \dots, b_{2m-4}, b_{2m-2}) = (1, 2, \dots, m-1, \frac{3^{m+1} + 2m - 3}{4}).$$

In order to show Propositions 6.1 and 6.2, by Theorems 1.5 and 4.2 we need merely consider the case of $d = 1$ and calculate the (mod 2) Betti numbers of $X_1(M, 2)$ in the following discussion.

6.1. The integral homology of $F_{\mathbb{Z}_2^2}(M, 2)$ for 2-dimensional small covers.

Let P be a ℓ -polygon with facets F_1, \dots, F_ℓ and vertices v_1, \dots, v_ℓ , as shown in the following diagram:



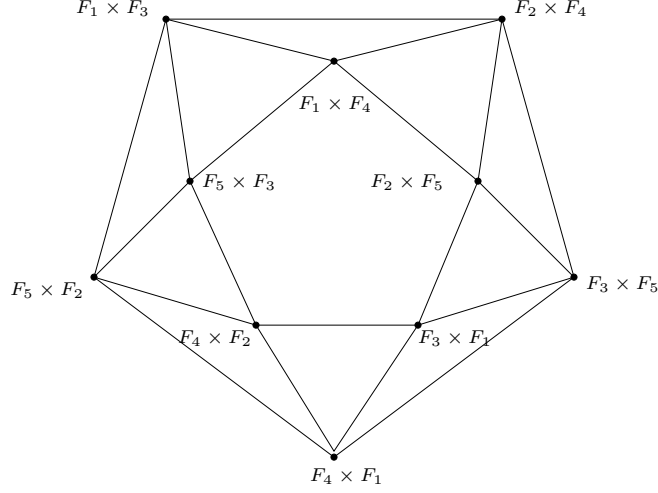
By Theorem 4.2, $F_{\mathbb{Z}_2^m}(M, 2)$ is homotopic to

$$X(\ell) = \begin{cases} \bigcup_{F_i \cap F_j = \emptyset} (\pi_1^{-1})^{\times 2}(F_i \times F_j) & \text{if } \ell > 3 \\ \bigcup_{v_i \cap F_j = \emptyset} (\pi_1^{-1})^{\times 2}(v_i \times F_j) \cup \bigcup_{F_i \cap v_j = \emptyset} (\pi_1^{-1})^{\times 2}(F_i \times v_j) & \text{if } \ell = 3. \end{cases}$$

Now let us determine the simplicial complex $K(\ell)$ dual to $\mathcal{A}(P, 2)$.

When $\ell = 3, 4$, it is easy to see that $K(3)$ is a 6-polygon with vertices $v_1 \times F_3, v_2 \times F_1, v_3 \times F_2, F_1 \times v_2, F_2 \times v_3, F_3 \times v_1$, and $K(4)$ is a 4-polygon with four vertices $F_1 \times F_3, F_2 \times F_4, F_3 \times F_1, F_4 \times F_2$. When $\ell = 5$, $K(5)$ is a 2-dimensional simplicial complex with 10 vertices. Actually $K(5)$ is exactly an annulus as shown

in the following diagram:



In general, when $\ell > 5$, $K(\ell)$ is a 3-dimensional simplicial complex with vertex set $\{F_i \times F_j \mid F_i \cap F_j = \emptyset\}$ such that there are 3-dimensional simplices of the form

$$\{F_i \times F_j, F_{i+1} \times F_j, F_i \times F_{j+1}, F_{i+1} \times F_{j+1}\}$$

where F_{i+1} will be F_1 if $i = \ell$, and F_{j+1} will be F_1 if $j = \ell$, and some additional 2-dimensional simplices of the form

$$\{F_i \times F_{i+2}, F_i \times F_{i+3}, F_{i+1} \times F_{i+3}\} \text{ or } \{F_{i+2} \times F_i, F_{i+3} \times F_i, F_{i+3} \times F_{i+1}\}$$

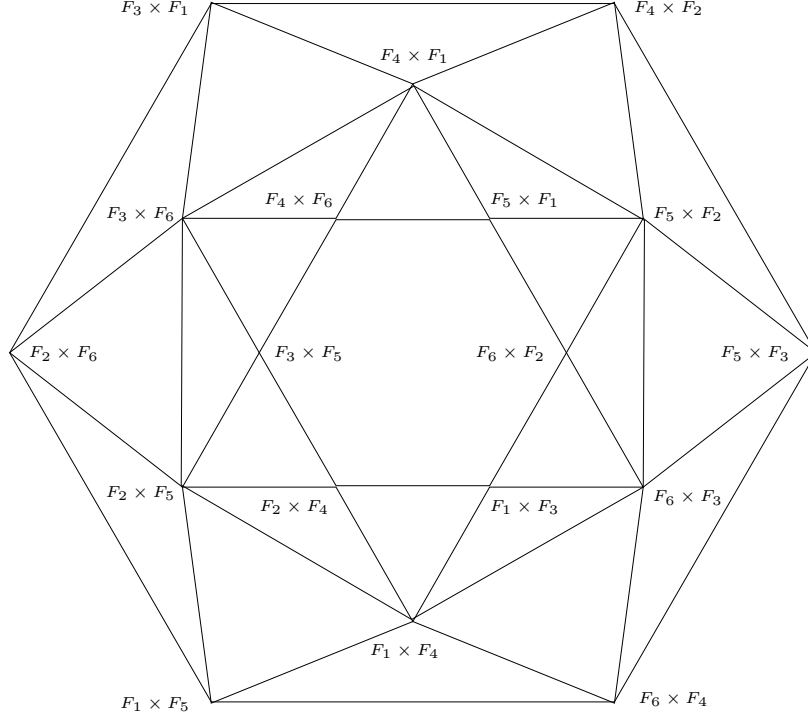
where F_{i+2} will be F_1, F_2 if $i = \ell - 1, \ell$ and F_{i+3} will be F_1, F_2, F_3 if $i = \ell - 2, \ell - 1, \ell$. Obviously, $K(\ell)$ contains $\ell(\ell - 3)$ vertices. We also know from Remark 7 that $K(\ell)$ is homotopic to a circle.

Next, for the convenience of calculation, let us choose a locally nice subcomplex $L(\ell)$ of $K(\ell)$ (for the notion of a locally nice subcomplex, see Definition 7.1). When $\ell \leq 5$, take $L(\ell) = K(\ell)$. When $\ell \geq 6$, we take $L(\ell)$ in such a way that $L(\ell)$ contains \emptyset and all vertices of $K(\ell)$, and $2\ell(\ell - 4)$ 2-dimensional simplices of the following forms

$$\{F_i \times F_j, F_{i+1} \times F_j, F_{i+1} \times F_{j+1}\} \text{ and } \{F_i \times F_j, F_i \times F_{j+1}, F_{i+1} \times F_{j+1}\}.$$

In this case, it is easy to check that $L(\ell)$ is exactly an annulus, and it has $\ell(3\ell - 11)$ 1-simplices. For example, when $\ell = 6$, $L(6)$ is a 2-dimensional simplicial complex

as shown in the following picture:



With the above arguments together, we have

Lemma 6.1. *When $\ell \leq 4$, $L(3)$ is a 6-polygon and $L(4)$ is a 4-polygon. When $\ell \geq 5$, $L(\ell)$ is a triangulation of an annulus with $\ell(\ell - 3)$ vertices, $\ell(3\ell - 11)$ 1-simplices and $2\ell(\ell - 4)$ 2-simplices.*

Now, to complete the proof of Proposition 6.1, it suffices to show the following result.

Proposition 6.3. *$X(3)$ is a 1-dimensional connected CW complex with Betti numbers $(b_0, b_1) = (1, 7)$. When $\ell \geq 4$, $X(\ell)$ is a 2-dimensional connected CW complex with Betti numbers $(b_0, b_1, b_2) = (1, 2\ell + 1, \ell(\ell - 3))$.*

Proof. Each vertex of $L(\ell)$ is of the form $F_i \times F_j$ if $\ell > 3$, and of the form $v_i \times F_j$ or $F_i \times v_j$ if $\ell = 3$. Since each $\pi_1^{-1}(F_i)$ is a circle and $\pi_1^{-1}(v_i)$ is a point, we have that $(\pi_1^{-1})^{\times 2}(F_i \times F_j)$ is a torus, and $(\pi_1^{-1})^{\times 2}(v_i \times F_j)$ (or $(\pi_1^{-1})^{\times 2}(F_i \times v_j)$) is also a circle. Also, for all $\ell \geq 3$, $L(\ell)$ is connected. Thus, $X(3)$ is a 1-dimensional connected CW complex and when $\ell \geq 4$, $X(\ell)$ is a 2-dimensional connected CW complex.

Now we first have by Remark 11 and Lemma 6.1 that $E_{p,0}^2(L(\ell)) = H_p(S^1)$, so $E_{p,0}^2(L(\ell)) = 0$ if $p > 1$ and $E_{0,0}^2(L(\ell)) \cong E_{1,0}^2(L(\ell)) \cong \mathbb{Z}$.

If $\ell = 3$, it is easy to see that $E_{p,q}^1(L(3)) = 0$ for $p > 0$ and $q > 1$, and $E_{0,1}^1(L(3)) \cong \mathbb{Z}^6$. So, we have that

$$E_{p,q}^2(L(3)) = \begin{cases} 0 & \text{if either } p > 1 \text{ and } q = 0 \text{ or } p > 0 \text{ and } q > 1 \\ \mathbb{Z} & \text{if } p \leq 1 \text{ and } q = 0 \\ \mathbb{Z}^6 & \text{if } p = 0 \text{ and } q = 1 \end{cases}$$

so $E^\infty(L(3)) = E^2(L(3))$.

If $\ell > 3$, then we have that $E_{p,q}^1(L(\ell)) = 0$ for either $p > 1$ and $q > 2$ or $p > 0$ and $q = 2$. By a direct calculation, we obtain that $E_{0,2}^2(L(\ell)) \cong \mathbb{Z}^{\ell(\ell-3)}$, $E_{0,1}^1(L(\ell)) \cong \mathbb{Z}^{2\ell(\ell-3)}$ and $E_{1,1}^1(L(\ell)) \cong \mathbb{Z}^{2\ell(\ell-4)}$. When $\ell = 4$, $\mathbb{Z}^{2\ell(\ell-4)}$ means the trivial group 0, and so $E_{1,1}^2(L(4)) = 0$ and $E_{0,1}^2(L(4)) \cong \mathbb{Z}^8$. When $\ell > 4$, it is easy to check that

$$0 \longrightarrow E_{1,1}^1(L(\ell)) \longrightarrow E_{0,1}^1(L(\ell)) \longrightarrow 0$$

is isomorphic to the simplicial complex given by the disjoint union of 2ℓ copies of a segment, so $E_{1,1}^2(L(\ell)) = 0$ and $E_{0,1}^2(L(\ell)) \cong \mathbb{Z}^{2\ell}$. Thus, if $\ell > 3$,

$$E_{p,q}^2(L(\ell)) = \begin{cases} 0 & \text{if either } p > 1 \text{ and } q = 0 \text{ or } p > 0 \text{ and } q > 2 \\ \mathbb{Z} & \text{if } p \leq 1 \text{ and } q = 0 \\ \mathbb{Z}^{2\ell} & \text{if } p = 0 \text{ and } q = 1 \\ \mathbb{Z}^{\ell(\ell-3)} & \text{if } p = 0 \text{ and } q = 2 \end{cases}$$

so $E^\infty(L(\ell)) = E^2(L(\ell))$.

Furthermore, we may conclude the desired Betti numbers for $X(\ell)$ by Theorem 7.1. \square

6.2. The case in which P is an m -simplex Δ^m with $m > 1$. Let $sd(Bd(\Delta^m))$ be the barycentric subdivision of the boundary complex of Δ^m , which is an $(m-1)$ -dimensional simplicial complex. Each simplex of $sd(Bd(\Delta^m))$ will be expressed as the form $\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k$ where σ_i is a face of $Bd(\Delta^m)$ (so $0 \leq \dim \sigma_i \leq m-1$), and is understood as a vertex of $sd(Bd(\Delta^m))$. Let i, j be non-negative integers with $i + j + 1 \leq m$. By $K_{i,j}^m$ we denote the subcomplex of $sd(Bd(\Delta^m))$, formed by those simplices $\{\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k \mid \dim \sigma_1 \geq i, \dim \sigma_k \leq m - j - 1\}$. It is easy to check that $K_{i,j}^m$ has the following properties:

- $K_{i,j}^m$ is $(m - i - j - 1)$ -dimensional and connected.
- For $i' \leq i$ and $j' \leq j$, $K_{i,j}^m \subseteq K_{i',j'}^m$.
- $K_{0,j}^m$ is the barycentric subdivision of the $(m - j - 1)$ -dimensional skeleton of $Bd(\Delta^m)$. In particular, $K_{0,0}^m = Bd(\Delta^m)$.

Lemma 6.2. $K_{i,j}^m$ is combinatorially equivalent to $K_{j,i}^m$.

Proof. This follows by mapping each vertex σ of $K_{i,j}^m$ to the vertex $\bar{\sigma}$ of $K_{j,i}^m$ and each simplex $\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k$ to $\bar{\sigma}_k \subset \cdots \subset \bar{\sigma}_2 \subset \bar{\sigma}_1$, where the $\bar{\sigma}$ means the complement of σ in the boundary complex $Bd(\Delta^m)$ of Δ^m , i.e., $\bar{\sigma}$ is the face of Δ^m , determined by those vertices which are not contained in σ . \square

Lemma 6.3. When $m = i + j + 1$, $H_r(K_{i,j}^m) = 0$ if $r \neq 0$, and when $m > i + j + 1$, $H_r(K_{i,j}^m) = 0$ if $r \neq 0$ and $r \neq m - i - j - 1$.

Proof. When $m = i + j + 1$, $K_{i,j}^m$ is a 0-dimensional complex, so $H_r(K_{i,j}^m) = 0$ if $r \neq 0$.

Now suppose that $m > i + j + 1$. Given a simplex $\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k$ in $K_{i,j}^m$, if $\dim \sigma_k < m - j - 1$, then it belongs to $K_{i,j+1}^m$. If $\dim \sigma_k = m - j - 1$, then it is easy to see that this simplex $\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k$ belongs to the following subcomplex of $K_{i,j}^m$

$$\bigcup_{\substack{\sigma_l \in K_{i,j}^m \\ \dim \sigma_l = m - j - 1}} \overline{St(\sigma_l, K_{i,j}^m)}.$$

Also, $Lk(\sigma_k, K_{i,j}^m) = \{\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_s \mid \dim \sigma_1 \geq i, \sigma_s \subset \sigma_k \text{ and } \sigma_s \neq \sigma_k\}$, and obviously it is the subcomplex $K_{i,0}^{m-j-1}$ of $sd(Bd(\sigma_k))$. Thus, we conclude that

$$(6.1) \quad K_{i,j}^m = K_{i,j+1}^m \bigcup \bigcup_{\substack{\sigma_l \in K_{i,j}^m \\ \dim \sigma_l = m-j-1}} \overline{St(\sigma_l, K_{i,j}^m)}$$

and for $\sigma_l \in K_{i,j}^m$ with $\dim \sigma_l = m-j-1$

$$(6.2) \quad \overline{St(\sigma_l, K_{i,j}^m)} \cap K_{i,j+1}^m = Lk(\sigma_l, K_{i,j}^m) = K_{i,0}^{m-j-1}.$$

Claim B. For $m > i+j+1$, $H_s(K_{i,j}^m, K_{i,j+1}^m) = 0$ if $s \neq m-i-j-1$.

Using the axiom of excision and (6.1)–(6.2), we see that

$$H_s(K_{i,j}^m, K_{i,j+1}^m) \cong \bigoplus_{\substack{\sigma_l \in K_{i,j}^m \\ \dim \sigma_l = m-j-1}} H_s(\overline{St(\sigma_l, K_{i,j}^m)}, Lk(\sigma_l, K_{i,j}^m)).$$

For $s > 1$, since

$$H_s(\overline{St(\sigma_l, K_{i,j}^m)}, Lk(\sigma_l, K_{i,j}^m)) = H_{s-1}(Lk(\sigma_l, K_{i,j}^m)) = H_{s-1}(K_{i,0}^{m-j-1})$$

and $K_{i,0}^{m-j-1}$ has the same homology as the $(m-i-j-2)$ -skeleton of $Bd(\sigma_l)$, it follows that Claim B holds.

We note that $K_{0,0}^m$ is just $sd(Bd(\Delta^m))$, so $H_r(K_{0,0}^m) = 0$ if $r \neq 0$ and $r \neq m-1$. Consider the long exact sequence

$$\cdots \rightarrow H_k(K_{i,j+1}^n) \rightarrow H_k(K_{i,j}^n) \rightarrow H_k(K_{i,j}^n, K_{i,j+1}^n) \rightarrow H_{k-1}(K_{i,j+1}^n) \rightarrow \cdots$$

Moreover, using an induction and Claim B, we may easily obtain the required result. \square

Now let M be a small cover over an m -dimensional simplex Δ^m . Then by Theorem 4.2, $F_{\mathbb{Z}_2^m}(M, 2)$ is homotopic to

$$X_1(M, 2) = \bigcup_{\sigma_i \in \mathcal{F}(\Delta^m)} (\pi_1^{-1})^{\times 2}(\sigma_i \times \overline{\sigma_i}).$$

Obviously, $X_1(M, 2)$ is an $(m-1)$ -dimensional CW complex. In order to apply Mayer-Vietoris spectral sequence, we choose a locally nice complex L in such a way that the vertex set of L consists of all $(\pi_1^{-1})^{\times 2}(\sigma_i \times \overline{\sigma_i})$ where $\sigma_i \in \mathcal{F}(\Delta^m)$, and each simplex of L is of the form

$$[(\pi_1^{-1})^{\times 2}(\sigma_{i_1} \times \overline{\sigma_{i_1}}), \dots, (\pi_1^{-1})^{\times 2}(\sigma_{i_r} \times \overline{\sigma_{i_r}})]$$

with $\sigma_{i_1} \subset \cdots \subset \sigma_{i_r}$. If we map $(\pi_1^{-1})^{\times 2}(\sigma_i \times \overline{\sigma_i})$ to σ_i , we see that L is combinatorially equivalent to $sd(Bd(\Delta^m))$. With this understood, we will identify L with $sd(Bd(\Delta^m))$.

Proof of Proposition 6.2. Given a simplex a of the form $\sigma_{i_1} \subset \cdots \subset \sigma_{i_r}$ in L , it is easy to see that

$$X_a = (\pi_1^{-1})^{\times 2}(\sigma_{i_1} \times \overline{\sigma_{i_1}}) \cap \cdots \cap (\pi_1^{-1})^{\times 2}(\sigma_{i_r} \times \overline{\sigma_{i_r}}) = (\pi_1^{-1})^{\times 2}(\sigma_{i_1} \times \overline{\sigma_{i_r}}).$$

It is well-known that M is homeomorphic to $\mathbb{R}P^m$ and for each face σ of Δ^m , $\pi_1^{-1}(\sigma)$ is homeomorphic to $\mathbb{R}P^{\dim \sigma}$. Thus, X_a is actually homeomorphic to $\mathbb{R}P^s \times \mathbb{R}P^t$ where $s = \dim \sigma_{i_1}$ and $t = \dim \overline{\sigma_{i_r}}$. Moreover,

$$H_k(X_a; \mathbb{Z}_2) \cong H_k(\mathbb{R}P^s \times \mathbb{R}P^t; \mathbb{Z}_2) = \sum_{i+j=k} H_i(\mathbb{R}P^s; \mathbb{Z}_2) \otimes H_j(\mathbb{R}P^t; \mathbb{Z}_2)$$

is generated by $\beta^i \otimes \gamma^j$ with $i+j = k$, where β^i and γ^j are generators of $H_i(\mathbb{R}P^s; \mathbb{Z}_2)$ and $H_j(\mathbb{R}P^t; \mathbb{Z}_2)$. To emphasize $\beta^i \otimes \gamma^j$ as an element of $H_k(X_a; \mathbb{Z}_2)$, we shall denote it by $(\beta^i \otimes \gamma^j)_a$.

Define a map

$$f : E_{*,q}^1(L) = \bigoplus_{a \in L} H_q(X_a; \mathbb{Z}_2) \longrightarrow \bigoplus_{i+j=q} \mathcal{C}_*(K_{i,j}^m; \mathbb{Z}_2)$$

by mapping $(\beta^i \otimes \gamma^j)_a$ to $a \in K_{i,j}^m$, where $\mathcal{C}_*(K_{i,j}^m; \mathbb{Z}_2)$ is the chain complex of $K_{i,j}^m$ with \mathbb{Z}_2 coefficients. Then f is a chain map since the boundary operator on $E_{*,q}^1(L)$ agrees with the boundary operator on $\bigoplus_{i+j=q} \mathcal{C}_*(K_{i,j}^m; \mathbb{Z}_2)$ by Remark 11. It is easy to check that f is a bijection, so f is actually a chain isomorphism. Thus, we obtain that

$$(6.3) \quad E_{p,q}^2(L) \cong \bigoplus_{i+j=q} H_p(K_{i,j}^m; \mathbb{Z}_2).$$

Note that if $q = 0$, then $K_{0,0}^m = sd(Bd(\Delta^m)) = L$, so $E_{p,0}^2(L) \cong H_p(S^{m-1}; \mathbb{Z}_2)$. This is also shown in Remark 11 since each X_a is connected.

Now by Lemma 6.3 and (6.3), we have that $E_{p,q}^2(L) = 0$ if $p+q \neq m-1$ and $p \neq 0$. Also it is easy to see that $\dim E_{0,q}^2(L) = q+1$ if $q \leq m-1$.

Combining the above arguments, we have that if $p+q \neq m-1$, then $E_{p,q}^\infty(L) = E_{p,q}^2(L)$, so $\dim H_k(X_1(M, 2); \mathbb{Z}_2) = k+1$ for $k < m-1$.

Now by Hopf trace theorem, we have that

$$\sum_{k=0}^{m-1} (-1)^k \dim H_k(X_1(M, 2); \mathbb{Z}_2) = \sum_{k=0}^{m-1} (-1)^k \dim D_k(X_1(M, 2); \mathbb{Z}_2)$$

where $D_k(X_1(M, 2); \mathbb{Z}_2)$ is the k -dimensional cellular chain group of $X_1(M, 2)$. By the construction of small covers, it is easy to check that

$$\dim D_k(X_1(M, 2); \mathbb{Z}_2) = 2^k \sum_{i+j=k} \binom{m+1}{i+1} \binom{m-i}{j+1}$$

so

$$\begin{aligned} & \dim H_{m-1}(X_1(M, 2); \mathbb{Z}_2) \\ &= \sum_{k=0}^{m-1} (-1)^k \dim D_k(X_1(M, 2); \mathbb{Z}_2) - \sum_{k=0}^{m-2} (-1)^k \dim H_k(X_1(M, 2); \mathbb{Z}_2) \\ &= \sum_{k=0}^{m-1} (-2)^k \sum_{i+j=k} \binom{m+1}{i+1} \binom{m-i}{j+1} - \sum_{k=0}^{m-2} (-1)^k (k+1) \\ &= \frac{3^{m+1} + 2m - 3}{4}. \end{aligned}$$

This completes the proof. \square

7. APPENDIX–MAYER-VIETORIS SPECTRAL SEQUENCE

Suppose that X is a CW-complex with all cells indexed by J , and X_1, \dots, X_N are subcomplexes of X such that $\bigcup_i X_i = X$ and all possible intersections of X_1, \dots, X_N are subcomplexes of X . Associated with X , we may define an abstract simplicial

complex K (including empty set) with vertices $1, \dots, N$ (or X_1, \dots, X_N) as follows: if $X_{i_1} \cap \dots \cap X_{i_r} \neq \emptyset$, then $\{i_1, \dots, i_r\} \in K$. For each $a \in K$, we define

$$X_a = \begin{cases} \bigcap_{i \in a} X_i & \text{if } a \neq \emptyset \\ X & \text{if } a = \emptyset. \end{cases}$$

Set $D_{p,q}^K(X) = \bigoplus_{a \in K, |a|=p+1} D_q(X_a)$ where $D_*(X_a) = \{D_q(X_a)\}$ is the cellular chain complex of X_a . Then we shall see that $D_{*,*}^K(X)$ has a natural double complex structure.

Let e_α be a cell of X in $\{e_\alpha | \alpha \in J\}$. Define $K(e_\alpha) = \{a \in K \mid e_\alpha \subset X_a\}$. Obviously, $K(e_\alpha)$ is a subcomplex determined by some simplex of K , so it is acyclic. If $a \in K(e_\alpha)$, then e_α would be a generator of $D_{\dim e_\alpha}(X_a)$, denoted by $e_{\alpha,a}$. Furthermore, we may write each cellular chain of $D_{p,q}^K(X) = \bigoplus_{a \in K, |a|=p+1} D_q(X_a)$ as

$$\sum_{\alpha \in J(q)} \sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} e_{\alpha,a}$$

where $k_{\alpha,a} \in \mathbb{Z}$, and $J(q)$ means that for $\alpha \in J(q)$, $\dim e_\alpha = q$.

Let $c = \sum_{a \in K(e_\alpha)} \lambda_a a$ be a chain in the simplicial chain complex $\mathcal{C}_*(K(e_\alpha))$ of $K(e_\alpha)$. Define $e_{\alpha,c} = \sum_{a \in K(e_\alpha)} \lambda_a e_{\alpha,a}$. Then it is easy to check that

Lemma 7.1. $e_{\alpha,c} = 0$ if and only if $c = 0$.

Now two differentials on $D_{*,*}^K(X)$ are defined as follows: One is $\partial_1 : D_{p,q}^K(X) \rightarrow D_{p,q-1}^K(X)$ given by

$$\partial_1 \left(\sum_{\alpha \in J(q)} \sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} e_{\alpha,a} \right) = \sum_{\alpha \in J(q)} \sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} \partial(e_{\alpha,a})$$

which is induced by the boundary homomorphism ∂ of $D_*(X_a)$, and the other one is $\partial_2 : D_{p,q}^K(X) \rightarrow D_{p-1,q}^K(X)$ given by

$$(7.1) \quad \partial_2 \left(\sum_{\alpha \in J(q)} \sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} e_{\alpha,a} \right) = \sum_{\alpha \in J(q)} \sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} e_{\alpha, \partial' a}$$

which is induced by the boundary homomorphism ∂' of the simplicial chain complex $\mathcal{C}_*(K(e_\alpha))$. Note that for the empty set $\emptyset \in K$, $\partial' \emptyset = 0$.

It is easy to check that $\partial_1 \partial_2 = \partial_2 \partial_1$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \partial_2 \uparrow & & \partial_2 \uparrow & & \partial_2 \uparrow & \\
0 \xleftarrow{\partial_1} & D_{-1,0}^K(X) & \xleftarrow{\partial_1} & D_{-1,1}^K(X) & \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_1} & D_{-1,q}^K(X) & \xleftarrow{\partial_1} \cdots \\
& \partial_2 \uparrow & & \partial_2 \uparrow & & \partial_2 \uparrow & \\
0 \xleftarrow{\partial_1} & D_{0,0}^K(X) & \xleftarrow{\partial_1} & D_{0,1}^K(X) & \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_1} & D_{0,q}^K(X) & \xleftarrow{\partial_1} \cdots \\
& \partial_2 \uparrow & & \partial_2 \uparrow & & \partial_2 \uparrow & \\
\vdots & \vdots & & \vdots & & \vdots & \vdots \\
& \partial_2 \uparrow & & \partial_2 \uparrow & & \partial_2 \uparrow & \\
0 \xleftarrow{\partial_1} & D_{p,0}^K(X) & \xleftarrow{\partial_1} & D_{p,1}^K(X) & \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_1} & D_{p,q}^K(X) & \xleftarrow{\partial_1} \cdots \\
& \partial_2 \uparrow & & \partial_2 \uparrow & & \partial_2 \uparrow & \\
\vdots & \vdots & & \vdots & & \vdots & \vdots
\end{array}$$

Now, let us look at the structure of this double complex $(D_{*,*}^K(X), \partial_1, \partial_2)$.

Proposition 7.1. *Every column of the above diagram is exact, i.e., for each q ,*

$$0 \xleftarrow{\partial_2} D_{-1,q}^K(X) \xleftarrow{\partial_2} D_{0,q}^K(X) \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_2} D_{p,q}^K(X) \xleftarrow{\partial_2} \cdots$$

is exact.

Proof. Suppose that $\sum_{\alpha \in J(q)} \sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} e_{\alpha,a}$ is a cycle in $D_{p,q}^K(X)$. Then

$$\partial_2 \left(\sum_{\alpha \in J(q)} \sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} e_{\alpha,a} \right) = 0.$$

Furthermore, we have that for each $\alpha \in J(q)$, $\partial_2(\sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} e_{\alpha,a}) = 0$. By Lemma 7.1, we obtain that for each $\alpha \in J(q)$, $\partial'(\sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} a) = 0$, so $\sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} a$ is a cycle in $\mathcal{C}_p(K(e_\alpha))$. Since $K(e_\alpha)$ is acyclic, there exists a chain c_α in $\mathcal{C}_{p+1}(K(e_\alpha))$ such that $\partial' c_\alpha = \sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} a$, so $\sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} e_{\alpha,a} = e_{\alpha, \sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} a} = e_{\alpha, \partial' c_\alpha} = \partial_2(e_{\alpha, c_\alpha})$. Therefore, we conclude that

$$\sum_{\alpha \in J(q)} \sum_{\substack{a \in K(e_\alpha) \\ |a|=p+1}} k_{\alpha,a} e_{\alpha,a} = \partial_2 \left(\sum_{\alpha \in J(q)} e_{\alpha, c_\alpha} \right)$$

is also a boundary chain, as desired. \square

Define $E_{p,q}^1(K)$ as the q -th homology group of the p -th row in the above diagram for $p \geq 0$, and when $p < 0$, let $E_{p,q}^1(K) = 0$. Let $\text{Im}_{p,q}^K \partial_2 = \text{Im}(D_{p,q}^K(X) \xrightarrow{\partial_2} D_{p-1,q}^K(X))$. Then we have the induced chain complex:

$$0 \longleftarrow \text{Im}_{p,0}^K \partial_2 \longleftarrow \text{Im}_{p,1}^K \partial_2 \longleftarrow \text{Im}_{p,2}^K \partial_2 \longleftarrow \cdots \longleftarrow \text{Im}_{p,q}^K \partial_2 \longleftarrow \cdots$$

Let $A_{p,q}(K)$ be the q -th homology group of this chain complex. Note that when $p < 0$, let $A_{p,q}(K) = A_{0,p+q}(K)$.

Because every column in the above diagram is exact, we have the following short exact sequence:

$$0 \longrightarrow \text{Im}_{p+1,*}^K \partial_2 \longrightarrow D_{p,*}^K(X) \xrightarrow{\partial_2} \text{Im}_{p,*}^K \partial_2 \longrightarrow 0$$

Furthermore, we may obtain the following long exact sequence:

$$\cdots \xrightarrow{i} A_{p+1,q}(K) \xrightarrow{j} E_{p,q}^1(K) \xrightarrow{k} A_{p,q}(K) \xrightarrow{i} A_{p+1,q-1}(K) \xrightarrow{j} E_{p,q-1}^1(K) \xrightarrow{k} \cdots$$

Then we have the exact couple $A(K) \xrightarrow{i} A(K)$, and the spectral

$$\begin{array}{ccc} A(K) & \xrightarrow{i} & A(K) \\ & \nwarrow k & \nearrow j \\ & E(K) & \end{array}$$

sequence $E_{p,q}^1(K), E_{p,q}^2(K), \dots, E_{p,q}^\infty(K)$.

By the theory of spectral sequence, we have that

Theorem 7.1.

$$H_i(X) = \bigoplus_{p+q=i} E_{p,q}^\infty(K)$$

Remark 11. We can explicitly write out the differential $j \circ k$ of the chain complex

$$\cdots \xrightarrow{j \circ k} E_{p,q}^1(K) \xrightarrow{j \circ k} E_{p-1,q}^1(K) \xrightarrow{j \circ k} \cdots \xrightarrow{j \circ k} E_{0,q}^1(K) \longrightarrow 0.$$

At first, for $a \in K$, given an element $\beta_a \in H_*(X_a)$. Let $b \subset a$. Then we define $\beta_{a,b}$ as the image of β_a under the map $H_*(X_a) \rightarrow H_*(X_b)$ induced by the nature imbedding $X_a \hookrightarrow X_b$. Now if $c = \sum_i \lambda_i b_i$ is a chain of the simplicial chain complex of K where $b_i \subset a$, define $\beta_{a,c}$ as $\sum_i \lambda_i \beta_{a,b_i}$. Next, by the definition of $E(K)$, we know that $E_{p,q}^1(K) = \bigoplus_{\substack{a \in K \\ |a|=p+1}} H_q(X_a)$, so we can write its element as $\sum_{\substack{a \in K \\ |a|=p+1}} \beta_a$. Moreover, we see easily that

$$j \circ k \left(\sum_{\substack{a \in K \\ |a|=p+1}} \beta_a \right) = \sum_{\substack{a \in K \\ |a|=p+1}} \beta_{a, \partial' a}.$$

In particular, when $q = 0$ and every X_a is connected, the chain complex

$$\cdots \xrightarrow{j \circ k} E_{p,0}^1(K) \xrightarrow{j \circ k} E_{p-1,0}^1(K) \xrightarrow{j \circ k} \cdots \xrightarrow{j \circ k} E_{0,0}^1(K) \longrightarrow 0$$

is isomorphic to the simplicial chain complex of $K \setminus \{\emptyset\}$. Thus, $E_{p,0}^2(K)$ is isomorphic to $H_p(K \setminus \{\emptyset\})$. Note that since we have assumed that $\emptyset \in K$, $H_p(K)$ is isomorphic to the reduced homology $\tilde{H}_p(K \setminus \{\emptyset\})$.

Generally, K may have a very complicated structure. This will lead to a difficulty for calculating the spectral sequence $E_{p,q}^1(K), E_{p,q}^2(K), \dots, E_{p,q}^\infty(K)$ induced by the double complex $D_{*,*}^K(X)$. For the purpose of our application, we shall choose a suitable subcomplex of K , so that this may give a simpler calculation.

Definition 7.1. A subcomplex L of K is said to be *locally nice* if L satisfies the following properties:

- L contains all vertices of K and the empty set \emptyset .
- For each cell e_α of X , $L(e_\alpha) = \{a \in L \mid e_\alpha \subset X_a\}$ is acyclic.

Now let L be a locally nice subcomplex of K . Similarly, we can define a double complex $D_{*,*}^L(X) = \{D_{p,q}^L(X)\}$, where $D_{p,q}^L(X) = \bigoplus_{a \in L, |a|=p+1} D_q(X_a)$. Then we see that for each q ,

$$0 \xleftarrow{\partial_2} D_{-1,q}^L(X) \xleftarrow{\partial_2} D_{0,q}^L(X) \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_2} D_{p,q}^L(X) \xleftarrow{\partial_2} \cdots$$

is still exact since $L(e_\alpha) = \{a \in L | e_\alpha \subset X_a\}$ is acyclic for each cell e_α of X . Thus, we can induce a corresponding spectral sequence $E_{p,q}^1(L), E_{p,q}^2(L), \dots, E_{p,q}^\infty(L)$ such that

$$H_i(X) = \bigoplus_{p+q=i} E_{p,q}^\infty(L).$$

It should be pointed out that there is also a cohomological version of the above argument. Namely, we can obtain a spectral sequence $E_1^{p,q}(L), E_2^{p,q}(L), \dots, E_\infty^{p,q}(L)$ from (X, L) such that

$$H^i(X) = \bigoplus_{p+q=i} E_\infty^{p,q}(L).$$

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SCHOOL OF MATHEMATICAL SCIENCES, FUDAN UNIVERSITY, SHANGHAI, 200433, CHINA
E-mail address: 072018012@fudan.edu.cn

SCHOOL OF MATHEMATICAL SCIENCES, FUDAN UNIVERSITY, SHANGHAI, 200433, CHINA
E-mail address: zlu@fudan.edu.cn

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, SINGAPORE 119260,
 REPUBLIC OF SINGAPORE
E-mail address: matwujie@math.nus.edu.sg
URL: <http://www.math.nus.edu.sg/~matwujie>